A chance-constrained optimisation model for the pricing and ordering omnichannel problem

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1 Introduction

An increasing number of enterprises in the retailing industry are now advancing to omnichannel configurations and adopting modern innovations to integrate their stores to provide clients with a comprehensive buying experience. In this context, we propose and study an optimization problem for an omnichannel model based on a price-setting single-period newsvendor model. This model decides the selling price and order quantity for each channel. Given the total market stochastic demand, each channel's demand is determined through an attractive model (a function of all the selling prices across channels).

The proposed model is applicable to seasonal products as well as products with short life cycles. It is also applicable to perishable products when units carried on the shelf are all of the same age. We focus on the relationship between expected profit and the service level for a price-setting omnichannel model. This relationship is important because the cycle service level (CSL) is frequently used by inventory managers as a metric to measure business quality and set the inventory policy, rather than the shortage cost to represent the economic consequence of a stock out.

The optimization problem is to maximize the total expected profit. The inequality constraints are set initially on the CSL - the probability that the demand does not exceed the order quantity. Thus, it is a chance-constrained problem. This problem is solvable since the decision variables and randomness can be decoupled. In addition, the constraints can be transformed into deterministic constraints using probability density functions. Therefore, nonlinear programming approaches can be used to solve the problem. However, given the inequality constraints, there are numerous difficulties. Most of which are related to structure, stability of transforming chance constraints into deterministic constraints, joint concavity of the objective function, convergence of numerical methods to determine the solution. In short, finding its optimal solution via chance-constrained programming or some well-known algorithms for solving deterministic non-linear constraints optimization problem is challenging and out of scope for this paper.

In this paper, we study a relaxed problem with equality constraints on CSL. It is still a non-linear chance-constrained problem. It is also complicated to find the optimal solution. Despite its challenges, we prove that the problem is mathematically well-behaved. Precisely, we can reduce the number of decision variables, investigate an equivalent unconstrained problem, and verify the joint concavity of its objective function. Moreover, there exists a unique solution to the first-order conditions. This solution corresponds to the maximization equality chance-constrained problem. As a consequence, a simple numerical method can be applied to find this point.

The rest of the paper is structured as follows. The next section represents the problem formulation. The third section investigates the optimization problem with non-linear equality constraints on CSL. The last section reports and discusses the results.

2 Optimization problem

Let us consider the following problem : a retailer buys a type of product from suppliers and distributes them to consumers through n channels $I = \{1, 2, \dots, n\}$. Each channel's demand depends on the total market demand and all the other channels demand via an attraction function of prices. Its corresponding profit is then determined via a price-dependent newsvendor model, with the constraint that the related order quantity satisfies a constraint on CSL. Thus, the retailing (or total) profit is the sum of all the channels' profit. In this paper, our goal is to find the optimal prices and order sizes to maximize the expected total profit.

$\boldsymbol{Q} = (Q_1, Q_2, \dots, Q_n)$ is the vector of order sizes, $\boldsymbol{c} = (c_1, c_2, \dots, c_n)$ is the vector of purchase costs per unit, $\boldsymbol{s} = (s_1, s_2, \dots, s_n)$ is the vector of salvage values for unsold units left at the end of the period, $c_i \geq s_i \geq 0$, $\boldsymbol{r} = (r_1, r_2, \dots, r_n)$ is the vector of selling prices, $\boldsymbol{D} = (D_1, D_2, \dots, D_n)$ is the vector of demand, ξ is the total market demand, characterized by its PDF $f_\xi(x)$ and CDF $F_\xi(x)$. $g_i(r_i)$ is the attraction function of customers to channel i,

$$D_i(r) = \xi \frac{g_i(r_i)}{g_0 + \sum_{k=1}^{n} g_k(r_k)}. \tag{1}$$

This model is similar to the Luce selection model (Luce 1959). For the motivation of using attractive function to modelize omnichannel demand, we refer it to [4]. Here, more general than the omnichannel demand suggested by [4], we consider that the total market demand, ξ, is stochastic with known distribution. In our model, $g_i(r_i)$ can be understood as a positive measure of the attractiveness of channel i, and the expected demand for each channel is proportional to the channel's attractiveness. Furthermore, the attractiveness of the no-purchase option might be interpreted as g_0. In literature, there are some frequently used attraction models : linear attraction model : $g_i(r_i) = a_i - b_i r_i$, with $a_i > 0, b_i > 0, \min a_i - b_i r_i > 0$; multinomial logit (MNL) model : $g_i(r_i) = \exp(a_i - b_i r_i)$ with $a_i > 0, b_i > 0$; multiplicative competitive interaction (MCI) model : $g_i(r_i) = a_i r_i^{-b_i}$ with $a_i > 0$, $b_i > 1$.

Let $G_i(\boldsymbol{r}) = \frac{g_i(r_i)}{g_0 + \sum_{k \in I} g_k(r_k)}$. Thus, it represents the proportion of the total demand sharing in channel i,

$$D_i(\boldsymbol{r}) = \xi G_i(\boldsymbol{r}). \tag{2}$$

In our paper, we employ the newsvendor model to determine the profit for each selling channel ([5]). At the end of the selling season, the actual profit for the retailer from channel i is

$$\Pi_i \left(D_i, Q_i, r_i\right) = r_i \min(D_i, Q_i) + s_i (Q_i - D_i)^+ - c_i Q_i. \tag{3}$$

The retailer wants to optimize his profit. However, the retailer is unable to determine the actual end of selling period profit because the demand was not realized at the start of the selling season. As a result, the classical method to solve the optimization problem is to assume a risk-averse retailer who makes the best pricing decision at the start of the sales season in order to maximize overall expected profit.

The expected profit for channel $i \in I$,

$$\mathbb{E}[\Pi_i \left(D_i, Q_i, r_i\right)] = \mathbb{E}[\Pi_i \left(\xi G_i(\boldsymbol{r}), Q_i, r_i\right)] := \Pi_i \left(Q_i, \boldsymbol{r}\right), \tag{4}$$

is a function of corresponding channel order quantity and all the retailing prices. The explicit formula is given in the Proposition 1.

Proposition 1. *The expected profit for channel $i \in I$ is as follows*

$$\begin{aligned} \Pi_i \left(Q_i, \boldsymbol{r}\right) = (r_i - s_i) G_i(\boldsymbol{r}) \mathbb{E}[\xi] - (c_i - s_i) Q_i \\ - (r_i - s_i) G_i(\boldsymbol{r}) \int_{\frac{Q_i}{G_i(\boldsymbol{r})}}^{\infty} \left(x - \frac{Q_i}{G_i(\boldsymbol{r})}\right) f_\xi(x) dx. \end{aligned} \tag{5}$$

The total expected profit is given as follows

$$\Pi(\boldsymbol{Q}, \boldsymbol{r}) = \sum_{i \in I} \Pi_i (Q_i, \boldsymbol{r}). \tag{6}$$

Here, $\Pi(\boldsymbol{Q}, \boldsymbol{r})$ is concave in each r_i (Q_i), given the values of all $r_j, j \neq i$ and $\boldsymbol{Q}$ ($Q_j, j \neq i$ and $\boldsymbol{r}$). However, it is not available to verify if $\Pi(\boldsymbol{Q}, \boldsymbol{r})$ is jointly concave on all its variables (all $(Q_i)_{i\in I}$ and $(r_i)_{i\in I}$). Thus, the (unconstrained) problem of maximizing $\Pi(\boldsymbol{Q}, \boldsymbol{r})$ includes difficulties (algorithmic, computational approaches) on finding global solutions.

In practice, the retailers not only want to maximize their profit, but they also ask for a high level of the business quality. The problem thus become maximizing $\Pi(\boldsymbol{Q}, \boldsymbol{r})$, under some managerial requirements. In this paper, the CSL measure is considered since it is frequently used by retailers to evaluate the quality of their selling process. Recall that the cycle service level related to channel $i \in I$ is the probability that the demand does not exceed the order quantity $\mathbb{P}(D_i \leq Q_i)$. Let $CSL_i^m \in [0, 1]$ be the minimum service level related to channel i specified by the retailer. Thus, the inequality constraint on CSL for channel i is $\mathbb{P}(D_i \leq Q_i) \geq CSL_i^m$.

Therefore, the optimization problem with inequality constraints is follows

$$
\begin{aligned}
\max_{\boldsymbol{Q},\boldsymbol{r}} \quad & \Pi(\boldsymbol{Q}, \boldsymbol{r}) \\
\text{s.t.} \quad & \mathbb{P}(D_i \leq Q_i) \geq CSL_i^m, \forall i \in I, \\
& \boldsymbol{r} \geq \boldsymbol{c}, \\
& \boldsymbol{Q} \geq \boldsymbol{0}.
\end{aligned}
\tag{P}
$$

Problem (P) is a chance-constrained problem. In 1959, Charnes and Cooper ([3]) presented chance-constrained programming as a method for solving optimization problems in the face of uncertainty. They handled the problem by proposing a methodology for ensuring that a model's decision resulted in a certain probability of constraint compliance. The method is reviewed and developed to increase efficiency (see Abdel, 2020 [2]). Problem (P) is solvable since the decision variables and randomness can be decoupled. However, there are numerous difficulties. Most of which are related to structure, stability of transforming chance constraints into deterministic constraints, joint concavity of the objective function, convergence of numerical methods to determine the maximal solution.

Problem (P) can be interpreted as a non-linear optimization problem under inequality constraints since its constraints on CSL can be transformed into deterministic constraints using probability density function. Precisely, let $h_i(Q_i, \boldsymbol{r}) := F_\xi(Q_i/G_i(\boldsymbol{r})) - CSL_i^m$, and $\boldsymbol{h}(\boldsymbol{Q}, \boldsymbol{r}) := (h_1(Q_1, \boldsymbol{r}), h_2(Q_2, \boldsymbol{r}), \ldots, h_n(Q_n, \boldsymbol{r}))$. Then, the inequality constraints on CSL can be rewritten as $\boldsymbol{h}(\boldsymbol{Q}, \boldsymbol{r}) \geq \boldsymbol{0}$. Indeed, there are some methods for solving this type of problem. Interior Point or Sequential Quadratic Programming algorithm can handle nonlinear equality and inequality constraints very tightly and give us a certificate of local optimality. However, they require to provide a reasonably good initial guess, the objective function and constraints should not include any discontinuity, and they only converge to local optima.

In short, problem (P) is complicated to solve. Finding its optimal solution via chance-constrained programming or some well-known algorithms for solving deterministic non-linear constraints optimization problem is challenging and out of scope for this paper.

In the next section, despite the complication, we study a relaxed version of problem (P). We show that, given the equality chance constraints, the objective function only depends on retailing price variables (no order quantity variables anymore). Moreover, we prove that it is jointly concave on the vector of prices.

3 On the joint concavity of the relaxed problem

In this section, we consider a relaxed version for the problem (P) by considering the equality constraints, $\boldsymbol{h}(\boldsymbol{Q}, \boldsymbol{r}) = \boldsymbol{0}$. We investigate an equivalent problem where the constraints are embedded in the objective function. We show that under a set of conditions, the resulting objective function is jointly concave in prices. Thus, there exists a unique solution to the first order condition and it is the optimal solution to the relaxed problem. This solution is a local candidate for the initial problem (P).

Remind that the decision variables and the randomness can be decoupled and the constraints can be relaxed into deterministic constraints using probability distribution functions. Moreover, based on the equality constraint, the decision variables can be reduced from $(\boldsymbol{Q}, \boldsymbol{r})$ to $\boldsymbol{r}$. Indeed, from $\boldsymbol{h}(\boldsymbol{Q}, \boldsymbol{r}) = \boldsymbol{0}$, it follows that for all $i \in I$,

$$Q_i = G_i(\boldsymbol{r}) F_\xi^{-1} \left(CSL_i^m\right). \tag{7}$$

The following proposition represents the objective function for the relaxed problem as a function of price variables $(r_i)_{i \in I}$ (no order quantity variables).

Proposition 2. *Given the equality constraints (7), the relaxed problem's objective function (6) is represented as follows*

$$\hat{\Pi}(\boldsymbol{r}) = \sum_{i \in I} \Big\{ G_i(\boldsymbol{r}) \Big[(r_i - c_i) F_\xi^{-1}(CSL_i^m) - (r_i - s_i) CSL_i F_\xi^{-1}(CSL_i^m) + (r_i - s_i) \int_{-\infty}^{F_\xi^{-1}(CSL_i^m)} x f_\xi(x) dx \Big] \Big\}. \tag{8}$$

Let $\Omega = [c_1, \infty) \times [c_2, \infty) \times \cdots \times [c_n, \infty)$ be the feasible domain for $\boldsymbol{r}$, our problem is to find the maximum of $\hat{\Pi}(\boldsymbol{r})$ given in (8),

$$\max_{\boldsymbol{r} \in \Omega} \quad \hat{\Pi}(\boldsymbol{r}). \tag{Q}$$

It is challenging to decide the concavity of the objective function of problem (Q) since it is not possible to check if its associated Hessian matrix is negative semi-definite at every point in Ω. Despite the difficulties, we claim that the function $\hat{\Pi}(\boldsymbol{r})$ given in (8) is jointly concave via a novel approach, given a set of assumptions as follows.

Assumptions

A_1. $g_i(r_i)$ is strictly decreasing in r_i.

A_2. $\mathbb{E}[\xi] < \infty$, $\lim_{r_i \to \infty} g_i(r_i) = 0$, $\lim_{r_i \to \infty} r_i g_i(r_i) = 0$.

A_3. $\lim_{r_i \to \infty} \zeta_i(r_i) > -\infty$, $\zeta_i(r_i) - \dfrac{\zeta_i'(r_i)}{\zeta_i(r_i)} < 0$, where $\zeta_i(r_i) := \dfrac{g_i'(r_i)}{g_i(r_i)}, \forall i \in I$.

B_1. ξ is a non-negative random variable.

B_2. At boundary point of prices $\boldsymbol{c}$, the total expected profit is non-decreasing in all the prices, $\nabla_{\boldsymbol{r}} \hat{\Pi}(\boldsymbol{c}) \geq \boldsymbol{0}$.

Remark 1.

i. Assumption A_1 says that the attractiveness of a channel is decreasing in the channel's price. This assumption implies that $G_i(\boldsymbol{r})$ is decreasing in r_i and increasing in r_j for $i \neq j$, which can be interpreted as the expected demand for channel i is decreasing in its own retailing price and increasing in the retailing prices of other channels.

ii. Assumption A_2 guarantees that the expected demand is finite and a channel's contribution becomes zero as its price becomes arbitrarily large.

iii. Assumption A_3 is rather technical and not very restrictive. For instance, the assumption is satisfied for linear, MCI, MNL attraction models.

iii. Assumption B_1 means that the total market demand is non-negative. If the support of ξ contains negative part, its truncated version can be used to restrict the domain. However, it is not indispensable from a technical point of view. Moreover, when B_1 is satisfied, Q_i is obviously non-negative for all $i \in I$.

iv. Assumption B_2 is not only reasonable in management point of view, but it also secures the existence of such a point $\hat{\boldsymbol{r}}$ satisfying $\nabla_{\boldsymbol{r}} \hat{\Pi}(\hat{\boldsymbol{r}}) = \boldsymbol{0}$ (solution to the first order condition). To our best understanding through experiments, the solution to the first order condition exists without condition B_2., but no analytical proof is available.

To prove the joint concavity, the normal approach by verifying the negative-semi definite property of the Hessian matrix for arbitrary point in the domain is not possible. We use a different approach through a relation between the zero-point of a function and its concavity. The lemma below represents the mentioned property.

Lemma 1. *Given a real number a, let $\psi : [a, \infty) \to \mathbb{R}$ be a $\mathcal{C}^2$ function of the single variable u. Suppose that*

i. There exists at least one point $\bar{u} \in [a, \infty)$ such that $\psi'(\bar{u}) = 0$,

ii. At any $\bar{u}$ that satisfies $\psi'(\bar{u}) = 0$, $\psi''(\bar{u}) < 0$.

Then, there exists a unique u^* that satisfies $\psi'(u^*) = 0$, and u^* is the argument of the maxima of $\psi(\cdot)$.

Lemma 2. *Given a real value vector $\boldsymbol{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$, let $\Psi : W_1 \times W_2 \times \cdots \times W_n \to \mathbb{R}, W_i = [a_i, \infty) \subset \mathbb{R}$, be a $\mathcal{C}^2$ function of $\boldsymbol{u} = (u_1, u_2, \ldots, u_n) \in W_1 \times W_2 \times \cdots \times W_n$. For all $i \neq j$, and $i, j \in I = \{1, 2, \ldots, n\}$, suppose that*

i. There exists at least one point $\bar{\boldsymbol{u}}$ such that $\nabla\Psi(\bar{\boldsymbol{u}}) = \mathbf{0}$,

ii. At all the points $\bar{\boldsymbol{u}}$ mentioned above, $\dfrac{\partial^2\Psi}{\partial u_i^2}(\bar{\boldsymbol{u}}) < 0$ and $\dfrac{\partial^2\Psi}{\partial u_i \partial u_j}(\bar{\boldsymbol{u}}) = 0$ hold true.

Then, there exists a unique vector $\boldsymbol{u}^*$ that satisfies $\nabla\Psi(\boldsymbol{u}^*) = 0$, and $\boldsymbol{u}^*$ maximizes $\Psi(\cdot)$.

To simplify and shorten the formulations, for all $i \in I$, let us denote

i. $\alpha_i := F_\xi^{-1}(CSL_i^m)$,

ii. $A_i := \int_{-\infty}^{\alpha_i} x f_\xi(x) dx$,

iii. $U_i := \alpha_i(1 - CSL_i) + A_i$,

iv. $V_i(r_i) := \alpha_i\big[(r_i - c_i) - (r_i - s_i)CSL_i\big] + (r_i - s_i)A_i$.

With given vector CSL^m, α_i, A_i, and U_i are constants. $V_i(\cdot)$ is a function of r_i only. Furthermore, $\hat{\Pi}_i(\boldsymbol{r}) = G_i(\boldsymbol{r})V_i(r_i)$ and $\hat{\Pi}(\boldsymbol{r}) = \sum_{i \in I} G_i(\boldsymbol{r})V_i(r_i)$. The following remarks represent some related managerial and technical properties.

Remark 2. *For a general distribution of total market demand ξ,*

i. Price-elasticity of each channel's demand is satisfied when $\mathbb{E}[\xi] > 0$, $\zeta_i < 0$.

ii. Relation between order quantity and price : if $\alpha_i > 0$, then $\dfrac{\partial \hat{Q}_i}{\partial r_i}(\boldsymbol{r}) < 0$ and $\dfrac{\partial \hat{Q}_i}{\partial r_j}(\boldsymbol{r}) > 0$; if $\alpha_i < 0$, then $\dfrac{\partial \hat{Q}_i}{\partial r_i}(r) > 0$ and $\dfrac{\partial \hat{Q}_i}{\partial r_j}(r) < 0$.

iii. $\alpha_i > 0 \Leftrightarrow CSL_i > \mathbb{P}(\xi < 0)$.

iv. U_i is an increasing function in α_i and CSL_i if ξ is non-negative random variable.

Remark 3. *Suppose that B_1 holds. Assume further that $CSL_i > 0$ for all $i \in I$. We then have*

i. $\alpha_i > 0$

ii. $A_i > 0$

iii. $U_i > 0$

iv. $V_i(c_i) < 0$

v. $U_i < \alpha_i$

The proposition below shows the relationship between second-order derivatives, cross derivatives with first-order derivatives of the objective function of problem (Q). It is the key equations to prove its joint concavity property.

Proposition 3. *For all $i, j \in \{1, 2, \ldots, n\}, i \neq j$, we have*

i. $$\frac{\partial^2\hat{\Pi}}{\partial r_i^2}(\boldsymbol{r}) = \left[\zeta_i(r_i) - \frac{\zeta_i'(r_i)}{\zeta_i(r_i)}\right] G_i(\boldsymbol{r})U_i + \left[\frac{\zeta_i'(r_i)}{\zeta_i(r_i)} + \zeta_i(r_i)\left(1 - 2G_i(\boldsymbol{r})\right)\right] \frac{\partial\hat{\Pi}}{\partial r_i}(\boldsymbol{r}),$$

ii. $$\frac{\partial^2\hat{\Pi}}{\partial r_i \partial r_j}(\boldsymbol{r}) = -\left\{\zeta_j(r_j)G_j(\boldsymbol{r})\frac{\partial\hat{\Pi}}{\partial r_i}(\boldsymbol{r}) + \zeta_i(r_i)G_i(\boldsymbol{r})\frac{\partial\hat{\Pi}}{\partial r_j}(\boldsymbol{r})\right\}.$$

From Lemma 2 and Proposition 3, we obtain the structural results on the total expected profit function $\hat{\Pi}(\boldsymbol{r})$. They are represented in two propositions as follows

Proposition 4. *Suppose that assumptions A_1, A_2, A_3, B_1 hold. If there exists a point $\boldsymbol{r}^*$ satisfying $\nabla\Pi(\boldsymbol{r}^*) = \mathbf{0}$, it follows that $\dfrac{\partial^2\hat{\Pi}}{\partial r_i^2}(\boldsymbol{r}^*) < 0$ and $\dfrac{\partial^2\hat{\Pi}}{\partial r_i \partial r_j}(\boldsymbol{r}^*) = 0$, for all $i, j \in \{1, 2, \ldots, n\}, i \neq j$.*

Proposition 5. *Suppose that assumptions A_1, A_2, A_3, B_1, B_2 hold. Then, the function $\hat{\Pi}(\boldsymbol{r})$ is jointly concave in all prices $(r_i)_{i\in I}$. Moreover, there exists a unique point $\boldsymbol{r}^*$ satisfying $\nabla\hat{\Pi}(\boldsymbol{r}^*) = \mathbf{0}$. Let $\boldsymbol{Q}^* = (G_1(\boldsymbol{r}^*)F_{\xi}^{-1}(CSL_1^m), G_2(\boldsymbol{r}^*)F_{\xi}^{-1}(CSL_2^m), \ldots, G_n(\boldsymbol{r}^*)F_{\xi}^{-1}(CSL_n^m))$. Then, $(\boldsymbol{Q}^*, \boldsymbol{r}^*)$ is the unique global solution to the maximization problem*

$$\begin{aligned} \max_{\boldsymbol{Q},\boldsymbol{r}} \quad & \Pi(\boldsymbol{Q},\boldsymbol{r}) \\ s.t. \quad & \boldsymbol{h}(\boldsymbol{Q},\boldsymbol{r}) = \mathbf{0}, \\ & \boldsymbol{r} \geq \boldsymbol{c}, \\ & \boldsymbol{Q} \geq \mathbf{0}. \end{aligned} \tag{Pm}$$

To summary, in this section, we study a chance-constrained optimization problem under equality constraints on CSL (Pm), a relaxation of problem (P). It is equivalent to solving a non-linear maximization problem (Q). We prove that the objective function of problem (Q) is jointly concave in all its variables, given a set of assumptions. It follows that there exists unique solution derived from first-order condition.

4 Conclusions and perspectives

In this paper, we prove the joint concavity of the objective function of a chance-constrained optimization problem. This problem is motivated from omnichannel retailing when each channel's demand is a stochastic function of prices, and each channel's order quantity satisfies a constraint on retailer's cycle service level. Multiplicative attractive scenario was considered. Our result holds true whenever all the sharing demand functions satisfies price elasticity assumption. In addition, it also holds true whenever the total market demand' s probability distribution satisfy some regular conditions. The approach for verifying the joint concavity of the objective function is highlighted when the normal approach by checking the negative semi-definiteness of its associated Hessian matrix at all the points in the domain is not applicable.

For future research, one line of study is to investigate on chance-constrained programming and deterministic nonlinear programming to figure out the solution of problem (P). Another line for research is to analyze the optimization problem for a more extensive collection of demand models (for example, addictive demand) and other service' measure (such as fill rate). It would also be interesting to improve the omnichannel model, which is embedded in the objective function, with multiple choice of ordering and dynamic pricing. Last but not least, it is meaningful to investigate the case of a decentralized supply chain and its effects on the objective function and the optimal solutions.

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