Approximating SDP solutions with linear programs and simplex-like algorithm

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1 Introduction

The scope of this paper is to compute tight upper bounds of standard semidefinite programs of the form :

$$(SDP)\begin{cases} \sup\langle C, X\rangle \\ \text{s.t.} \\ \quad \langle A_i, X\rangle = b_i\,, \quad i = 1, ..., m \\ \quad X \succeq 0 \\ \quad X \in \mathcal{S}^n \end{cases} \tag{1}$$

Where C, A_i for $i = 1, \dots, m$ are symmetric $n \times n$ matrices, $\mathcal{S}^n$ is the cone of symmetric matrices of dimension n and $\mathcal{S}^n_+$ the cone of positive semidefinite matrices.

Semidefinite programming has been widely studied and numerous algorithms have been designed to solve (SDP) [1]. However, in this paper, we focus on analyzing an approach leading to tight upper bounds using linear programming.

2 Hierarchy of linear programs

The underlying idea of the approach we intend to study was published in [2]. One considers an easier version of (1) by replacing the semidefiniteness constraint by a more tractable one, i.e., X being *diagonally dominant* (resp. *scaled diagonally dominant*). The resulting problem is an LP (resp. SOCP). In the following, we describe an iterative method that approximate the optimal solution of (1) by using diagonally dominant matrices and by changing basis at each iteration. This approach was first published in [3].

First, note that the dual to (1) is given by

$$\begin{aligned} &\inf\, y^T b \\ &\text{s.t. } C - \sum_{i=1}^m y_i A_i \succeq 0\,. \end{aligned} \tag{2}$$

Consider the following DD strengthening for an SDP of the form (2) with optimal value d^*

$$\begin{aligned} d_0^* = &\inf\, y^T b \\ &\text{s.t. } \sum_{i=1}^m y_i A_i - C \text{ is } DD. \end{aligned} \tag{3}$$

Let $\tilde{Y} = \sum_{i=1}^m y_i A_i - C$ be an optimal solution with objective value $d_0^* \geq d^*$. Then there exists a matrix P such that $P^T \tilde{Y} P$ is a *diagonal* matrix. Thus, consider the second iteration

$$\begin{aligned} d_1^* = \inf\ & y^T b \\ \text{s.t. } & P^T \left(\sum_{i=1}^m y_i A_i - C \right) P \text{ is } DD \end{aligned} \tag{4}$$

Since there exists a feasible solution that is diagonal (the previous optimal solution), it readily follows that $d^* \leq d_1^* \leq d_0^*$. The only difference with the previous program is that, here we have a "better" basis P, meaning a basis such that $d_1 \leq d_0$.

Let now $\tilde{y}$ be optimal for (4) and compute a matrix Q such that $Q^T(\sum_{i=1}^m \tilde{y}_i A_i - C)Q$ is diagonal. Then $\tilde{y}$ is feasible for

$$\begin{aligned} d_2^* = \inf\ & y^T b \\ \text{s.t. } & Q^T \left(\sum_{i=1}^m y_i A_i - C \right) Q \text{ is } DD, \end{aligned} \tag{5}$$

and so $d^* \leq d_2^* \leq d_1^* \leq d_0^*$.

By continuing this hierarchy, we keep changing basis in order to improve the upper bounds for (1). We will present numerical experiments along with a discussion about convergence issues.

3 Conclusions and perspectives

In this abstract, we presented an algorithm to compute upper bounds to semidefinite programs thanks to a hierarchy of linear programs. At each iteration, we find an optimal basis for the next iteration as it is the case in the simplex algorithm for the linear case. Our aim is to analyze the numerical properties and convergence behavior of this method.

Références

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