Parameterized complexity of a single machine scheduling problem

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Many scheduling problems, even seemingly basic ones such as $P2|chains|C_{max}$, have been proved (strongly) NP-hard [7] over the years. [4] even showed that this problem remains weakly NP-hard with three chains. Upon reaching NP-hardness this quickly, it becomes difficult to establish anything deeper about these scheduling problems within the scope of classical complexity theory.

To answer this, parameterized complexity theory gives additional tools for a refined analysis of such hard scheduling problems. Given a parameter k and denoting n the input size, a problem is called $fixed-parameter\ tractable$ (FPT) parameterized by k if it can be solved in time $\mathcal{O}(poly(n) \times f(k))$ with f an arbitrary function [5]. The idea is to identify k as the limiting property and give a polynomial time algorithm for all instances with a bounded value of k. When the studied problem is believed to not be FPT, the W-hierarchy defined in [6] is usually used to evaluate its difficulty relative to the considered parameter. We have $FPT = W[0] \subseteq W[1] \subseteq W[2] \subseteq$ etc. and unless $P = NP$, all the inclusions are believed to be strict.

While parameterized complexity theory has been successfully applied to a lot of computer science areas, it has started to be studied extensively on scheduling only quite recently. One of the few older results was from [2] and showed that $P|prec, p_j = 1|C_{max}$ is W[2]-hard parameterized by the number m of machines. Some interesting parameters recently studied include the width w of the partial order giving the precedence constraints [4] and, in the case of a precedence graph only formed of chains, the number of chains c or the thickness τ, which is the maximum number of overlapping chain time windows at any given time [3].

The pathwidth - noted μ - which we define as the maximum number of overlapping job time windows at any given time, has been considered only very recently [1, 8, 9]. To the best of our knowledge, this parameter has had no hardness result relative to it proved yet. Currently known hardness results relative to other similar parameters such as the width w of the precedence graph do not match. For example [2] showed that $P2|prec, p_j \in \{1, 2\}|C_{max}$ is W[2]-hard parameterized by w, while it can be inferred from [8] that this problem is FPT parameterized by μ. With μ allowing such problems to be FPT when other parameters do not, it makes it all the more difficult to find a hardness result relative to μ.

We consider the decision problem $1|chains, r_i, \widetilde{d}_i, p_i = 1, l_{i,j}|\star$ with $l_{i,j}$ an exact delay (also called *time lag*) between two consecutive jobs in a chain. [3] established several hardness results with the number of chains c or the thickness τ as the parameter, but all the hardness reductions required delays to be unary in the input. We show that $1|chains, r_i, \widetilde{d}_i, p_i = 1, l_{i,j}|\star$ is W[1]-hard parameterized by μ. We reduce from INDEPENDENT SET which is a well-known W[1]-complete problem parameterized by the size k of the independent set. k chains of jobs are used to select the k nodes of the independent set and duplicate this information multiple times using exact delays. Then the edges are checked to not violate the supposedly independence of the chosen set. The edges are processed successively and individually in order to keep the pathwidth low. This reduction does not require unary delays and could potentially be reinvested in other scheduling problems involving delays.

Références

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