AC Optimal Power Flow: a strengthened SDP relaxation and an iterative MILP scheme for global optimization

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1 Introduction

The Alternating-Current Optimal Power Flow (ACOPF) is a fundamental optimization problem for power system analysis. Due to the nonconvexity of power flow equations, the ACOPF is a difficult problem both in theory and in practice. Thanks to Interior Point (IP) algorithms [11], developed starting in the 90s, the computation of ACOPF feasible solutions and local optima is accessible, even for instances of several thousand nodes [1]. During the last decade, researchers have been focusing on the design and solution of convex relaxations to bound the optimality gap of feasible solutions found by IP algorithms. A review of various relaxation techniques for the ACOPF problem is available in [8]. Despite these advances, finding global optima of ACOPF instances with several hundreds of buses is challenging for state-of-the-art algorithms, mostly branch-and-bound solvers based on Second-Order Cone Programming (SOCP) [6], Quadratic Convex (QC) [4] or Semidefinite Programming (SDP) [2] relaxations.

In this quest towards global optimality, our contribution is threefold: (i) we propose a new strengthening of the SDP relaxation based on angle difference limits on lines (ii) we combine feasibility and optimality bound-tightening methods with a dynamic programming algorithm to propagate the angles domain reduction (iii) we propose an original global optimization algorithm that proceeds by solving a sequence of dynamically generated Mixed-Integer Linear Programming (MILP) problems. Regarding the first point i.e. our new valid inequalities based on angle differences limits, we explain in Sect. 3.1 how they differ from the cuts proposed in [3, 9]. As concerns the third point: contrary to [7], a previous paper using piecewise relaxations for the ACOPF, we use MILP models with SDP cuts instead of Mixed-Integer QC models. We apply our global optimization algorithm on the IEEE PES PGLib [1] benchmark and compare the optimality gaps with other recent and state-of-the-art global optimization approaches [5, 9] that use this reference benchmark.

2 Mathematical programming formulations for the ACOPF

2.1 Original formulation

A power grid is as a network of buses interconnected by lines. It is modelled as an oriented graph $\mathcal{N} = (\mathcal{B}, \mathcal{L})$ with size $n := |\mathcal{B}|$. The set $\mathcal{L}$ is such that (s.t.) $\mathcal{L} \cap \mathcal{L}^R = \emptyset$, where $\mathcal{L}^R$ is the set of couples (b, a) s.t. $(a, b) \in \mathcal{L}$. A line $\ell \in \mathcal{L}$ is described by a couple (b, a) s.t. $b \in \mathcal{B}$ is the "from" bus (denoted by f), $a \in \mathcal{B}$ is the the "to" bus (denoted by t). Electricity generating units are located at several buses in the network. We denote by $\mathcal{G}_b$ the set of generators located at bus $b \in \mathcal{B}$. The set of all generators is $\mathcal{G} := \cup_{b \in \mathcal{B}} \mathcal{G}_b$, whose cardinality is denoted by $m := |\mathcal{G}|$. The parameters of the ACOPF problem are described in Table 1. The ACOPF is the nonconvex optimization problem formally described by formulation (**OPF**) below. We emphasize that for any complex number $x \in \mathbb{C}$, $x^* = \mathsf{Re}(x) - \mathbf{i}\,\mathsf{Im}(x)$ is its complex conjugate, $|x|$ is its magnitude and $\angle x$ its phase.

Parameters	Index set	Meaning
$c_{1g} \in \mathbb{R}, c_{2g} \in \mathbb{R}_+$	$g \in \mathcal{G}$	Generator's cost parameters
$\underline{s}_g, \overline{s}_g \in \mathbb{C}$	$g \in \mathcal{G}$	Generator's domain bounds
$\underline{v}_b, \overline{v}_b \in [0, 2]$	$b \in \mathcal{B}$	Normalised voltage magnitude bounds
$S^{\mathsf{d}}_b \in \mathbb{C}$	$b \in \mathcal{B}$	Power demand
$Y^s_b \in \mathbb{C}$	$b \in \mathcal{B}$	Shunt admittance
$Y^{\mathsf{ff}}_{ba}, Y^{\mathsf{ft}}_{ba}, Y^{\mathsf{tf}}_{ba}, Y^{\mathsf{tt}}_{ba} \in \mathbb{C}$	$(b, a) \in \mathcal{L}$	Line impedance matrix
$\underline{\theta}_{ba}, \overline{\theta}_{ba} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$	$(b, a) \in \mathcal{L} \cup \mathcal{L}^R$	Angle difference limits

TAB. 1: Parameters of the ACOPF problem

$$
\textbf{OPF}\begin{cases}
\min\limits_{V\in\mathbb{C}^n,\, S\in\mathbb{C}^m} & \sum\limits_{g\in\mathcal{G}} c_{1g}\,\mathsf{Re}(S_g) + c_{2g}\,\mathsf{Re}(S_g)^2 \\
\forall b \in \mathcal{B} & \underline{v}_b \le |V_b| \le \overline{v}_b \\
\forall g \in \mathcal{G} & \underline{s}_g \le S_g \le \overline{s}_g \\
\forall b \in \mathcal{B} & \sum\limits_{g\in\mathcal{G}_b} S_g = S^{\mathsf{d}}_b + (Y^s_b)^*|V_b|^2 \quad + \sum\limits_{(b,a)\in\mathcal{L}} (Y^{\mathsf{ff}}_{ba})^*|V_b|^2 + (Y^{\mathsf{ft}}_{ba})^* V_b V_a^* \\
& \qquad\qquad\qquad\qquad\qquad + \sum\limits_{(a,b)\in\mathcal{L}} (Y^{\mathsf{tt}}_{ab})^*|V_b|^2 + (Y^{\mathsf{tf}}_{ab})^* V_b V_a^* \\
\forall (b,a) \in \mathcal{L} & |(Y^{\mathsf{ff}}_{ba})^*|V_b|^2 + (Y^{\mathsf{ft}}_{ba})^* V_b V_a^*| \le \overline{S}_{ba} \\
\forall (a,b) \in \mathcal{L} & |(Y^{\mathsf{tt}}_{ab})^*|V_b|^2 + (Y^{\mathsf{tf}}_{ab})^* V_b V_a^*| \le \overline{S}_{ab} \\
\forall (b,a) \in \mathcal{L}\cup\mathcal{L}^R & \underline{\theta}_{ba} \le \angle V_b - \angle V_a \le \overline{\theta}_{ba}
\end{cases}
$$

2.2 ACOPF reformulation based on extended variables

We introduce a symmetric set $\mathcal{E} \subset \mathcal{B} \times \mathcal{B}$ of arcs s.t. $(\mathcal{B}, \mathcal{E})$ is a chordal extension of the network $\mathcal{N} = (\mathcal{B}, \mathcal{L})$, and a *clique tree* $\mathcal{T}$ of this chordal extension [10]. For any node $k \in \mathcal{T}$, the set $\mathcal{B}_k \subset \mathcal{B}$ is the corresponding clique of $(\mathcal{B}, \mathcal{E})$. We denote by $\mathbb{H}_n(\mathcal{E})$ the set of complex vectors W indexed by $\mathcal{E}$ and s.t. $W_{ba} = W^*_{ab}$ for all $(b, a) \in \mathcal{E}$. For any $k \in \mathcal{T}$, we denote by $\mathcal{B}_k$ the matrix $(W_{ba})_{(b,a)\in\mathcal{B}_k^2}$. Based on this notation, we reformulate the ACOPF as

$$
\textbf{OPF}_\mathbf{W}\begin{cases}
\min\limits_{W\in\mathbb{H}_n(\mathcal{E}),\, S\in\mathbb{C}^m} & \sum\limits_{g\in\mathcal{G}} c_{1g}\,\mathsf{Re}(S_g) + c_{2g}\,\mathsf{Re}(S_g)^2 \\
\forall b \in \mathcal{B} & \underline{v}_b^2 \le W_{bb} \le \overline{v}_b^2 \\
\forall g \in \mathcal{G} & \underline{s}_g \le S_g \le \overline{s}_g \\
\forall b \in \mathcal{B} & \sum\limits_{g\in\mathcal{G}_b} S_g = S^{\mathsf{d}}_b + (Y^s_b)^* W_{bb} \quad + \sum\limits_{(b,a)\in\mathcal{L}} (Y^{\mathsf{ff}}_{ba})^* W_{bb} + (Y^{\mathsf{ft}}_{ba})^* W_{ba} \\
& \qquad\qquad\qquad\qquad\qquad + \sum\limits_{(a,b)\in\mathcal{L}} (Y^{\mathsf{tt}}_{ab})^* W_{bb} + (Y^{\mathsf{tf}}_{ab})^* W_{ba} \\
\forall (b,a) \in \mathcal{L} & |(Y^{\mathsf{ff}}_{ba})^* W_{bb} + (Y^{\mathsf{ft}}_{ba})^* W_{ba}| \le \overline{S}_{ba} \\
\forall (a,b) \in \mathcal{L} & |(Y^{\mathsf{tt}}_{ab})^* W_{bb} + (Y^{\mathsf{tf}}_{ab})^* W_{ba}| \le \overline{S}_{ab} \\
\forall (b,a) \in \mathcal{L}\cup\mathcal{L}^R & \tan(\underline{\theta}_{ba})\mathsf{Re}(W_{ba}) \le \mathsf{Im}(W_{ba}) \le \tan(\overline{\theta}_{ba})\mathsf{Re}(W_{ba}) \\
\forall (a,b) \in \mathcal{E} & |W_{ba}|^2 = W_{bb} W_{aa} \qquad\qquad (\star) \\
\forall k \in \mathcal{T} & W_{\mathcal{B}_k,\mathcal{B}_k} \succeq 0
\end{cases}
$$

While the clique-based SDP relaxation is well known, this clique-based reformulation of the ACOPF problem itself is not properly stated in the literature, as far as we know. Yet, we acknowledge that the proof of Th. 1 is closely related to the developments presented in [2].

Theorem 1 *A pair (W, S) is feasible (resp. optimal) in* $(\mathbf{OPF_W})$ *if and only if it exists V s.t. (V, S) is a feasible (resp. optimal) solution of* $(\mathbf{OPF})$ *and $W_{ba} = V_b(V_a)^*$ for all $(b, a) \in \mathcal{E}$.*

Sketch of proof: We prove the equivalence for the feasibility, which also proves the equivalence for the optimality since both problems share the same objective value. We take (V, S) a feasible solution in $(\mathbf{OPF})$ and we define $W \in \mathbb{H}_n(\mathcal{E})$ as $W_{ba} = V_b V_a^*$ for any $(b, a) \in \mathcal{E}$. By substitution and noticing that $W_{\mathcal{B}_k,\mathcal{B}_k} = V_{\mathcal{B}_k} V^H_{\mathcal{B}_k} \succeq 0$, it is easy to see that (W, S) is feasible in $(\mathbf{OPF_W})$. Conversely, we take any couple (W, S) feasible in $(\mathbf{OPF_W})$. Since $W_{\mathcal{B}_k,\mathcal{B}_k} \succeq 0$ and $|W_{ba}|^2 = W_{bb}W_{aa}$ for all $(b, a) \in \mathcal{B}_k^2$, we can apply [2, Prop. 6] to state that $\mathsf{rank}\, W_{\mathcal{B}_k,\mathcal{B}_k} = 1$ for all $k \in \mathcal{T}$. Based on this and by induction on the number of cliques, we can construct a vector $V \in \mathbb{C}^n$ s.t. $W_{ba} = V_b(V_a)^*$ for all $(b, a) \in \mathcal{E}$ (as in [2, Prop. 7]). By substitution in the constraints of $(\mathbf{OPF_W})$, we see that (S, V) is feasible in $(\mathbf{OPF})$. □

3 Strenghtened SDP relaxation

In formulation ($\mathbf{OPF_W}$), the constraints ($\star$) are the only non-convex constraints. Removing them leads to the celebrated clique-based SDP relaxation [8]. Instead of deleting the constraints ($\star$), we outer-approximate them leveraging on the magnitude and angle difference bounds.

3.1 Convexification of the ($\star$) constraints

For all $b \in \mathcal{B}$, we introduce a variable $L_b \in [\underline{v}_b, \overline{v}_b]$ that represents the magnitude $|V_b|$. For all $(b,a) \in \mathcal{E}$, we introduce a variable $R_{ba} \in \mathbb{R}_+$ that stand for $|V_b||V_a|$ and is subject to

$$R_{ba} \geq \underline{v}_b L_a + \underline{v}_a L_b - \underline{v}_b \underline{v}_a \quad \wedge \quad R_{ba} \geq \overline{v}_b L_a + \overline{v}_a L_b - \overline{v}_b \overline{v}_a \tag{1}$$

$$R_{ba} \leq \overline{v}_b L_a + \underline{v}_a L_b - \underline{v}_a \overline{v}_b \quad \wedge \quad R_{ba} \leq \overline{v}_a L_b + \underline{v}_b L_a - \overline{v}_a \underline{v}_b. \tag{2}$$

For all $b \in \mathcal{B}$, we also apply the following constraints

$$L_b^2 \leq R_{bb} \quad \wedge \quad R_{bb} \leq \underline{v}_b^2 \frac{\overline{v}_b - L_b}{\overline{v}_b - \underline{v}_b} + \overline{v}_b^2 \frac{L_b - \underline{v}_b}{\overline{v}_b - \underline{v}_b} \quad \wedge \quad R_{bb} = W_{bb}. \tag{3}$$

Finally, for every clique $k \in \mathcal{T}$, we require that

$$\begin{pmatrix} 1 & L_{\mathcal{B}_k}^H \\ L_{\mathcal{B}_k} & R_{\mathcal{B}_k \mathcal{B}_k} \end{pmatrix} \succeq 0. \tag{4}$$

Whereas constraints (1)-(4) approximate the equality $R_{ba}^2 = W_{bb} W_{aa}$, we also need to approximate $|W_{ba}| = R_{ba}$. For this purpose, we add the following valid constraints for all $(b,a) \in \mathcal{E}$:

$$\cos(\frac{\underline{\theta}_{ba} + \overline{\theta}_{ba}}{2})\mathsf{Re}(W_{ba}) + \sin(\frac{\underline{\theta}_{ba} + \overline{\theta}_{ba}}{2})\mathsf{Im}(W_{ba}) \geq R_{ba} \cos(\frac{\overline{\theta}_{ba} - \underline{\theta}_{ba}}{2}) \quad \wedge \quad |W_{ba}| \leq R_{ba} \tag{5}$$

The following theorem show how the constraints (1)-(5) helps having $|W_{ba}|^2 \approx W_{bb} W_{aa}$ when the magnitude and angle bounds are tightened.

Theorem 2 *For $(b,a) \in \mathcal{E}$, we set $\Delta_1 := \max_{c \in \{b,a\}} \overline{v}_c - \underline{v}_c$ and $\Delta_2 = \overline{\theta}_{ba} - \underline{\theta}_{ba}$. Under constraints* (1)-(5) *and if $\Delta_1 \leq 1$, we have $||W_{ba}|^2 - W_{bb} W_{aa}| \leq 12\Delta_1^2 + 4\Delta_2^2$.*

For the sake of brevity, the proof of Theorem 2 is omitted in this short version of the article. Adding the decision variables L_b and R_{ba} to the optimization problem ($\mathbf{OPF_W}$) and replacing the constraints ($\star$) by the set of constraints (1)-(5), we obtain a conic programming problem ($\mathbf{R}$) in complex numbers, that is tighter than the rank-relaxation. This convexification approach differs from [3, 9] because we introduce the variables R_{ba} which are interlinked via McCormick (1) and SDP constraints (4), whereas the nonlinear cuts in the aforementioned articles independently convexify the feasible sets for pairs (W_{ba}, W_{bb}).

3.2 Feasibility-based bound tightening (FBBT)

The power flow limits in the lines implicitly imply bounds on the phase $\angle V_b V_a^*$ and thus could help us reduce the interval $[\underline{\theta}_{ba}, \overline{\theta}_{ba}]$ and thus the error bound in Th. 2. We choose any line $(b,a) \in \mathcal{L}$. Dividing the inequality $|(Y_{ba}^{\mathsf{ft}})^* V_b V_a^* + (Y_{ba}^{\mathsf{ff}})^* |V_b|^2| \leq \overline{S}_{ba}$ by $|Y_{ba}^{\mathsf{ft}} V_b V_a|$, we deduce that $|\frac{V_b V_a^*}{|V_a||V_b|} - z\frac{|V_b|}{|V_a|}| \leq R$, where $z := \frac{(Y_{ba}^{\mathsf{ff}})^*}{(Y_{ba}^{\mathsf{ft}})^*}$ and $R := \frac{S_\ell}{|Y_{ba}^{\mathsf{ft}} V_b V_a|}$. We notice that $u := \frac{V_b V_a^*}{|V_a||V_b|}$ is a unit complex number and has a nonnegative real part since $\angle V_b - \angle V_a \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Representing the ratio $\frac{|V_b|}{|V_a|}$ by a variable λ, we can formulate the following small convex programming problem

$$\max_{u,\lambda} \ \mathsf{Im}(u) \quad \text{s.t.}\ |u - z\lambda| \leq R \ \wedge \ \mathsf{Re}(u) \geq 0 \ \wedge \ |u| \leq 1 \ \wedge \ \lambda \in [\frac{\underline{v}_b}{\overline{v}_a}, \frac{\overline{v}_b}{\underline{v}_a}].$$

Denoting by $\overline{h}$ its value, we deduce that $\arcsin(\overline{h})$ is an upper-bound on $\angle V_b - \angle V_a$. Hence, we can redefine $\overline{\theta}_{ba} := \min(\overline{\theta}_{ba}, \arcsin(\overline{h}))$ without changing the value of ($\mathbf{OPF}$). If we minimize $\mathsf{Im}(u)$ under the same constraints to get a value $\underline{h}$, we can redefine $\underline{\theta}_{ba} := \max(\underline{\theta}_{ba}, \arcsin(\underline{h}))$. Similarly for any $(a,b) \in \mathcal{L}$, leveraging on $|(Y_{ab}^{\mathsf{tf}})^* V_b V_a^* + (Y_{ab}^{\mathsf{tt}})^* |V_b|^2| \leq \overline{S}_{ab}$, we can use the same procedure with $z := \frac{(Y_{ab}^{\mathsf{tt}})^*}{(Y_{ab}^{\mathsf{tf}})^*}$ and $R := \frac{S_\ell}{|Y_{ab}^{\mathsf{tf}} V_b V_a|}$ to tighten $\underline{\theta}_{ba}$ and $\overline{\theta}_{ba}$.

3.3 Optimality-based bound tightening (OBBT)

We use a local solver like an nonlinear IP solver to find an ACOPF feasible solution. The corresponding upper-bound $\overline{obj}$ enables us to add the constraint $\sum\limits_{g\in\mathcal{G}} c_{1g}\,\mathsf{Re}(S_g) + c_{2g}\,\mathsf{Re}(S_g)^2 \leq \overline{obj}$ to the relaxation $(\mathbf{R})$, which yields a convex set $\mathcal{F}$ for the tuplet (S, W, L, R). Then,

- (Magnitude) We set $\overline{v}_i \leftarrow \max_{(S,W,L,R)\in\mathcal{F}} L_i$ and $\underline{v}_i \leftarrow \min_{(S,W,L,R)\in\mathcal{F}} L_i$,
- (Phase) We compute $\overline{h}_{ba} := \max_{(S,W,L,R)\in\mathcal{F}} \mathsf{Im}(W_{ba})$ and $\underline{h}_{ba} := \min_{(S,W,L,R)\in\mathcal{F}} \mathsf{Im}(W_{ba})$ and set $\overline{\theta}_{ba} \leftarrow \min(\overline{\theta}_{ba}, \arcsin(\max(\frac{\overline{h}_{ba}}{\overline{v}_b\overline{v}_a}, \frac{\overline{h}_{ba}}{\underline{v}_b\underline{v}_a})))$ and $\underline{\theta}_{ba} \leftarrow \max(\underline{\theta}_{ba}, \arcsin(\min(\frac{\underline{h}_{ba}}{\overline{v}_b\overline{v}_a}, \frac{\underline{h}_{ba}}{\underline{v}_b\underline{v}_a})))$.

3.4 Floyd-Warshall algorithm to tighten angle difference bounds

Through FBBT and OBBT, we may individually improve the bounds $\underline{\theta}_{ba}$ and $\overline{\theta}_{ba}$ for any $(b, a) \in \mathcal{E}$. To propagate the bound tightening, we use a dynamic programming algorithm based on the following valid updates for any b, a, c s.t. $(b, a), (b, c), (c, a) \in \mathcal{E}^3$: $\overline{\theta}_{ba} \leftarrow \min(\overline{\theta}_{ba}, \overline{\theta}_{bc} + \overline{\theta}_{ca})$ and $\underline{\theta}_{ba} \leftarrow \max(\underline{\theta}_{ba}, \underline{\theta}_{bc} + \underline{\theta}_{ca})$. Hence, we notice that we can apply a Floyd-Warshall like algorithm. We execute it only inside cliques to avoid a computation burden.

4 A MILP-based global optimization algorithm

Leveraging on the solution of the strengthened SDP relaxation, we generate a sequence of MILP problems whose values converge to the ACOPF value.

4.1 Binary variables to encode piecewise linear constraints

Splitting magnitude intervals For any $b \in \mathcal{B}$, we may want to split the interval $[\underline{v}_b, \overline{v}_b]$ in K_b subintervals. We thus introduce pairs $(\underline{v}_{bk}, \overline{v}_{bk})$ for $k \in \{1, \ldots, K_b\}$ s.t. $\underline{v}_{bk} \leq \overline{v}_{bk}$, $\overline{v}_{bk} = \underline{v}_{b\,k+1}$, $\underline{v}_{b1} = \underline{v}_b$ and $\overline{v}_{bK_b} = \overline{v}_b$. For any $k \in \{1, \ldots, K_b\}$, we introduce a variable $x_{bk} \in \{0, 1\}$. To encode $x_{bk} = 1 \implies L_b \in [\underline{v}_{bk}, \overline{v}_{bk}]$, we write

$$\sum_{k=1}^{K_b} x_{bk} = 1 \quad \wedge \quad \sum_{k=1}^{K_b} \underline{v}_{bk} x_{bk} \leq L_b \leq \sum_{k=1}^{K_b} \overline{v}_{bk} x_{bk}.$$

Moreover we add the following constraint for every $k \in \{1, \ldots, K_b\}$,

$$R_{bb} \leq \underline{v}_{bk}^2 \frac{\overline{v}_{bk} - L_b}{\overline{v}_{bk} - \underline{v}_{bk}} + \overline{v}_{bk}^2 \frac{L_b - \underline{v}_{bk}}{\overline{v}_{bk} - \underline{v}_{bk}} + \overline{v}_b^2 (1 - x_{bx}).$$

For every $k \in \{1, K_b\}$ and for all $a \in \mathcal{B}$ s.t. $(b, a) \in \mathcal{E}$, we add the following inequalities:

$$R_{ba} \geq \underline{v}_{bk} L_a + \underline{v}_a L_b - \underline{v}_{bk}\underline{v}_a + \overline{v}_b\overline{v}_a(x_{bk} - 1) \quad \wedge \quad R_{ba} \geq \overline{v}_{bk} L_a + \overline{v}_a L_b - \overline{v}_{bk}\overline{v}_a + \overline{v}_b\overline{v}_a(x_{bk} - 1)$$
$$R_{ba} \leq \overline{v}_{bk} L_a + \underline{v}_a L_b - \underline{v}_a\overline{v}_{bk} + \overline{v}_b\overline{v}_a(1 - x_{bk}) \quad \wedge \quad R_{ba} \leq \overline{v}_a L_b + \underline{v}_{bk} L_a - \overline{v}_a\underline{v}_{bk} + \overline{v}_b\overline{v}_a(1 - x_{bk}).$$

Splitting angle intervals For any $(b, a) \in \mathcal{E}$, we may want to split the interval $[\underline{\theta}_{ba}, \overline{\theta}_{ba}]$ in J_{ba} subintervals. We introduce pairs $(\underline{\theta}_{baj}, \overline{\theta}_{baj})$ for $j \in \{1, \ldots, J_{ba}\}$ s.t. $\underline{\theta}_{baj} \leq \overline{\theta}_{baj}$, $\overline{\theta}_{baj} = \underline{\theta}_{ba\,j+1}$, $\underline{\theta}_{ba0} = \underline{\theta}_{ba}$ and $\overline{\theta}_{baJ_{ba}} = \overline{\theta}_{ba}$. For $j \in \{1, \ldots, J_{ba}\}$, we introduce a variable $\delta_{baj} \in \{0, 1\}$. To encode $\delta_{baj} = 1 \implies \angle W_{ba} \in [\underline{\theta}_{baj}, \overline{\theta}_{baj}]$, we write $\sum_{j=1}^{J_{ba}} \delta_{baj} = 1$ and for $j \in \{1, \ldots, J_{ba}\}$,

$$\tan(\underline{\theta}_{ba})\mathsf{Re}(W_{ba}) + (\delta_{baj} - 1)\overline{v}_b\overline{v}_a \leq \mathsf{Im}(W_{ba}) \leq \tan(\overline{\theta}_{ba})\mathsf{Re}(W_{ba}) + (1 - \delta_{baj})\overline{v}_b\overline{v}_a$$

For all $j \in \{1, \ldots, J_{ba}\}$, we also define the angle $\hat{\theta}_{baj} = \frac{\underline{\theta}_{baj} + \overline{\theta}_{baj}}{2}$ and write

$$\cos(\hat{\theta}_{baj})\mathsf{Re}(W_{ba}) + \sin(\hat{\theta}_{baj})\mathsf{Im}(W_{ba}) \geq R_{ba}\cos(\frac{\overline{\theta}_{baj} - \underline{\theta}_{baj}}{2}) + (\delta_{baj} - 1)\overline{v}_b\overline{v}_a$$

The division of the intervals $[\underline{v}_b, \overline{v}_b]$ and $[\underline{\theta}_{ba}, \overline{\theta}_{ba}]$ is dynamically made, by solving MILP problems of increasing size. In the following,"adding a magnitude breakpoint to b" means that $K_b \leftarrow K_b + 1$. Similarly,"adding an angle difference breakpoint to (b, a)" means $J_{ba} \leftarrow J_{ba} + 1$.

4.2 The MILP-based iterative scheme

The following asymptotically converging algorithm is executed for global optimization, based on (i) a local optimization IP solver (ii) a conic programming solver and (iii) a MILP solver.

0. **Initialization:** An ACOPF feasible solution is computed with a local solver and the conic programming relaxation $(\mathbf{R})$ is solved. If the gap is greater than targetOptGap, we apply FBBT and OBBT to $(\mathbf{R})$. Based on the optimal primal-dual solution of the conic problem $(\mathbf{R})$ we deduce active cuts for the QC and SDP constraints and generate a LP relaxation $(\mathbf{R_L})$ with the same value than $(\mathbf{R})$. The next iterations then tighten $(\mathbf{R_L})$.

1. **Outer-iterations:** As long as (i) the optimality gap is larger than targetOptGap and (ii) $\epsilon(W) := \max_{(b,a)\in\mathcal{E}} | |W_{ba}|^2 - W_{bb}W_{aa}| \geq \overline{\epsilon}$ (targeted accuracy), then:
   - For at most N_1 couples with largest violation $|R_{ba}^2 - W_{bb}W_{aa}| \geq \overline{\epsilon}$, we add magnitude breakpoints for b and a to divide the active subintervals $[\underline{v}_{bk}, \overline{v}_{bk}]$ and $[\underline{v}_{ak}, \overline{v}_{ak}]$,
   - For at most N_2 couples $(b, a) \in \mathcal{E}$ with largest violation $| |W_{ba}| - R_{ba}| \geq \overline{\epsilon}$, we add an angle difference breakpoint for (b, a) to divide the active subinterval $[\underline{\theta}_{baj}, \overline{\theta}_{baj}]$,
   - We solve the resulting MILP problem to global optimality to get (S, W, L, R).

2. **Inner-iterations:** While the solution (S, W, L, R) violate some QC and SDP constraints (within tolerance $\alpha\epsilon(W)$ for $\alpha \in]0, 1[$) of the relaxation $(\mathbf{R})$, we add the corresponding cuts and solve the resulting MILP problem to global optimality to get (S, W, L, R).

5 Numerical results

For all experiments, we use a 64-bit Ubuntu computer with 32 Intel(R) Xeon(R) CPU E5-2620 v4 @ 2.10GHz and 32 GB RAM. Along this algorithm, we use the commercial solvers MOSEK and CPLEX called through their Python APIs, as well as the academic solver MIPS [11]. Our code is available at `github.com/aoustry/SDP-MILP4OPF`. We consider an optimality gap of 0.01% for global optimality (**GOPT**) and we set a time limit of 10 hours for the bound-tightening and 5 hours for the iterative scheme. This study focuses on the network instances from the IEEE PES PGLib AC-OPF v21.07 library [1] with less than 300 buses. We compare our approach with the standard SOCP and SDP relaxations [8], and with two other global optimization approaches that have been applied to this benchmark. In [5], the authors compute the 1st and 2nd relaxations of Lasserre Hierarchy, as well as a SDP relaxation strengthened with RLT cuts and OBBT. In [9], the authors apply OBBT to several strengthened QC relaxations. Comparing the execution times would not necessarily be fair in so far as authors in [5, 9] use computing clusters of several machines, whereas we use a single machine; yet, we underline that our maximum execution time limit has the same order of magnitude as in [5, 9] i.e. between 5 and 15 hours. Table 2 shows that our approach closes the gap for 19 over 30 instances. For 7 of these 19 instances, it is the only one among the three benchmarked algorithms to reach global optimality. For 25 over 30 instances, our approach has the lowest optimality gap.

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References

[1] S. Babaeinejadsarookolaee, et al, "The Power Grid Library for Benchmarking AC Optimal Power Flow Algorithms," IEEE PES Task Force 2019.

[2] C. Chen, A. Atamtürk, and S. Oren, "A spatial branch-and-cut method for nonconvex QCQP with bounded complex variables. *Math. Prog.*," vol. 165, no. 2, pp. 549-577, 2017.

Case	AC Obj.	Optimality gap (%) SOCP	SDP	Best of [9]	Best of [5]	This work	Time (s) (**R**) & BT	MILPs
Typical Operating Condition (TYP)								
case3_lmbd	5.812e+03	1.32	0.39	0.01	**GOPT**	**GOPT**	3	<1
case5_pjm	1.755e+04	14.55	5.21	5.80	0.09	**GOPT**	9	122
case14_ieee	2.178e+03	0.11	**GOPT**	_	**GOPT**	**GOPT**	2	0
case24_ieee_rts	6.335e+04	0.02	**GOPT**	_	**GOPT**	**GOPT**	3	0
case30_as	8.031e+02	0.06	**GOPT**	_	**GOPT**	**GOPT**	5	0
case30_ieee	8.209e+03	18.84	**GOPT**	0.01	**GOPT**	**GOPT**	4	0
case39_epri	1.384e+05	0.56	**GOPT**	_	**GOPT**	**GOPT**	5	0
case57_ieee	3.759e+04	0.16	**GOPT**	_	**GOPT**	**GOPT**	7	0
case73_ieee_rts	1.898e+05	0.04	**GOPT**	_	**GOPT**	**GOPT**	12	0
case89_pegase	1.073e+05	0.75	0.37	_	0.29	**0.21**	TL	TL
case118_ieee	9.721e+04	0.91	0.07	0.02	0.03	**GOPT**	4910	3
case162_ieee_dtc	1.080e+05	5.95	1.77	**0.03**	0.45	0.86	34,620	TL
case179_goc	7.543e+05	0.16	0.07	_	0.07	**0.05**	6,550	TL
case240_pserc	3.330e+06	2.78	1.43	2.30	_	**1.01**	21,740	TL
case300_ieee	5.652e+05	2.63	0.71	0.06	0.10	**GOPT**	27,200	TL
Congested Operating Condition (API)								
case3_lmbd_api	1.124e+04	9.27	7.35	0.04	**GOPT**	**GOPT**	3	0
case5_pjm_api	7.638e+04	4.09	0.26	0.01	**GOPT**	**GOPT**	14	0
case14_ieee_api	5.999e+03	5.13	**GOPT**	0.01	**GOPT**	**GOPT**	2	0
case24_ieee_rts_api	1.349e+05	17.88	2.07	0.04	0.03	**GOPT**	570	0
case30_as_api	4.996e+03	44.61	2.06	0.72	0.39	**GOPT**	690	0
case30_ieee_api	1.804e+04	5.46	0.02	0.04	0.02	**GOPT**	680	0
case39_epri_api	2.497e+05	1.72	0.18	0.02	0.01	**GOPT**	1,060	1
case57_ieee_api	4.929e+04	0.08	**GOPT**	_	**GOPT**	**GOPT**	7	0
case73_ieee_rts_api	4.226e+05	12.87	2.90	0.41	0.75	**0.02**	6,220	TL
case89_pegase_api	1.302e+05	23.11	21.95	1.33	**0.93**	20.1	TL	TL
case118_ieee_api	2.422e+05	29.97	11.7	3.39	**0.99**	1.94	4,280	TL
case162_ieee_dtc_api	1.210e+05	4.36	1.42	**0.06**	0.26	0.57	34,320	TL
case179_goc_api	1.932e+06	9.88	0.55	**0.02**	0.54	0.53	12,780	TL
case240_pserc_api	4.641e+06	0.67	0.27	_	_	**0.18**	21,900	TL
case300_ieee_api	6.850e+05	0.85	0.09	_	0.07	**GOPT**	27,700	TL

TAB. 2: PGLIB v21.07 results

[3] C. Coffrin, H. Hijazi and P. van Hentenryck, "Strengthening the SDP relaxaion of AC power flows with convex envelopes, bound tightening, and valid inequalities," *IEEE Trans. Power Syst.*, vol. 32, no. 5, pp. 3549-3558, 2017.

[4] H. Godard, S. Elloumi, A. Lambert, J. Maeght and M. Ruiz. "Novel approach towards global optimality of optimal power flow using quadratic convex optimization," in *CoDIT*, pp. 1227–1232, 2019.

[5] S. Gopinath, H. Hijazi, T. Weisser, H. Nagarajan, M. Yetkin, K. Sundar, and R. Bent, "Proving global optimality of ACOPF solutions," *Electric Power Systems Research*, vol. 189, 2020.

[6] B. Kocuk, S. Dey and A. Sun, "Matrix minor reformulation and SOCP-based spatial branch-and-cut method for the AC optimal power flow problem," *Math. Prog.* vol. 10, pp. 557-–596, 2018.

[7] M. Lu, H. Nagarajan, R. Bent, S. Eksioglu, and S. Mason, "Tight piecewise convex relaxations for global optimization of optimal power flow," in *PSCC*, pp. 1–7, 2018.

[8] D. Molzahn and I. Hiskens, "A Survey of Relaxations and Approximations of the Power Flow Equations", *Found. Trends in Elec. Ener. Syst.*, vol. 4, no. 1-2, pp 1-221, 2019.

[9] K. Sundar, H. Nagarajan, S. Misra, M. Lu, C. Coffrin and R. Bent, "Optimization-based bound tightening using a strengthened QC-relaxation of the OPF problem," Tech. report, 2019.

[10] L. Vandenberghe, and M. S. Andersen, "Chordal graphs and semidefinite optimization," *Found. Trends Optim.*, vol. 1, no. 4, pp. 241-433, 2015.

[11] R. D. Zimmerman and H. Wang. "MATPOWER Interior Point Solver (MIPS) User's Manual, Version 1.4. 2020," Available: `https://matpower.org/docs/MIPS-manual-1.4.pdf`.