Robust Selection Problem with Decision-Dependent Information Discovery under Budgeted Uncertainty

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Keywords : *robust optimization, decision-dependent information discovery, selection problem, complexity*

1 Introduction

Robust optimization is a popular approach for modeling and solving optimization problems with uncertain data. Most studies in that field consider that the flow of information revealed during the decision stages is independent of the decision. Instead, here we consider that a part of the decision variables controls that flow. The *decision-dependent information discovery* (DDID) approach applies to many real-world applications, such as production planning where the production cost of a type of item is revealed after having taken the decision to produce it. The knowledge of that cost then contributes to deciding how many items of that type should be produced.

The research field of robust problems with DDID is recent, and mainly focuses on exact and approximate methods to solve them numerically, ie [4]. In this work, more focused on theory, we study the complexity of the robust selection problem with DDID under budgeted uncertainty. In particular, we present polynomial complexity results for two special cases of this problem and extend the second one to a broader class of problems.

2 Problem definition

Given n items and their nominal costs, the nominal selection problem aims to select p items minimizing the total cost. Let $I = \{1, \ldots, n\}$, c be the nominal cost vector, and $Y = \{y \in \{0,1\}^n : \sum_{i \in I} y_i = p\}$ be the set of feasible solutions to the problem. The robust selection problem under Γ-budgeted uncertainty considers that each item i has an uncertain cost $c_i + \delta_i d_i$, where δ is a deviation vector that can be any vector in $\Xi = \{\delta \in [0,1]^n : \sum_{i \in I} \delta_i \leq \Gamma\}$ [1]. In particular, this problem considers the worst-case deviation δ for any feasible solution in Y.

The robust selection problem with DDID under budgeted uncertainty considers the ability to choose b items to observe, lifting the uncertainty over their costs, before deciding on the selection. This problem can be formally defined as

$$z^* = \min_{w \in W} \max_{\gamma \in \Xi} \min_{y \in Y} \max_{\delta \in \Xi(w,\gamma)} \sum_{i \in I} y_i (c_i + d_i \delta_i), \qquad (\Gamma\text{−SPD})$$

where $W = \{w \in \{0,1\}^n : \sum_{i \in I} w_i = b\}$ and $\Xi(w, \gamma) = \{\delta \in \Xi : w_i \delta_i = w_i \gamma_i \ \forall i \in I\}$ [4]. In this model, the first min corresponds to the choice of items to observe, the first max considers the worst costs for the observed items, the second min selects items considering the observed costs, and finally the second max considers the worst costs for the selection, consistently with the observation.

3 $\Gamma-$SPD when $p = 1$

We consider $\Gamma-$SPD when $p = 1$. Then, the item selection amounts to the choice of one index $k \in I$.

$$z^* = \min_{w \in W} \max_{\gamma \in \Xi} \min_{k \in I} \max_{\delta \in \Xi(w,\gamma)} c_i + d_i \delta_i. \qquad (\Gamma-1\text{SPD})$$

We prove that $\Gamma-1$SPD can be solved with a tolerance of $\epsilon > 0$ in $\mathcal{O}(n \log \frac{n}{\epsilon})$ through a binary search on z^*. To that end, we can bound $z^* \in [\min_{k \in I} c_k, \min_{k \in I} c_k + d_k]$ and we consider the function

$$\Gamma(z) = \max_{x \in \{0,1\}^n} \left\{ \sum_{i \in I} \max\left\{0, \frac{z - c_i}{d_i}\right\} : \sum_{i \in I} x_i \leq b + 1 \right\}.$$

We show that for any $z \in [\min_{k \in I} c_k, \min_{k \in I} c_k + d_k]$, we have $z \leq z^*$ if and only if $\Gamma(z) \leq \Gamma$, which we use as a separation criterion for the binary search.

4 $\Gamma-$SPD when b is constant

We consider $\Gamma-$SPD when b is constant. Given $\tilde{w} \in W$, let $z^*(\tilde{w})$ be the optimal value of the outer maximization problem of $\Gamma-$SPD. Notice that $|W| < n^b$. Then, $\Gamma-$SPD can be solved in polynomial time by enumeration on W if and only if $z^*(\tilde{w})$ can be computed in polynomial time for each $\tilde{w} \in W$.

Given $\tilde{\gamma} \in \Xi$, let $z^*(\tilde{w}, \tilde{\gamma})$ be the optimal value of the inner minimization of $\Gamma-$SPD for $\tilde{w}$ and $\tilde{\gamma}$. Then, we consider the following hypograph reformulation to compute $z^*(\tilde{w})$.

$$z^*(\tilde{w}) = \max_{\gamma \in \Xi} \{\theta : \theta \leq z^*(\tilde{w}, \gamma)\}. \qquad (1)$$

Let F denotes the feasibility set of (1), we consider the following separation problem for F.

SEPARATION(F)
Input A vector $(\tilde{\theta}, \tilde{\gamma}) \in \mathbb{R} \times \mathbb{R}^n$.
Task Assert that $(\tilde{\theta}, \tilde{\gamma}) \in F$, or find a separating hyperplane, which means a vector $(a^\theta, a^\gamma) \in \mathbb{R} \times \mathbb{R}^n$ such that $a^\theta \theta + a^{\gamma\top} \gamma < a^\theta \tilde{\theta} + a^{\gamma\top} \tilde{\gamma}$ for all $(\theta, \gamma) \in F$.

We prove that SEPARATION(F) can be solved in polynomial time. Then, by the well-known equivalence between separation and optimization problems over a polyhedron [2], we prove the polynomial complexity of computing $z^*(\tilde{w})$ and solving $\Gamma-$SPD when b is constant

This result can be generalized to a broader class of robust problems with DDID. Let P be any problem of the form

$$\min_{w \in W} \max_{\zeta \in \Xi_{knap}} \min_{y \in Y} \max_{\xi \in \Xi_{knap}(w,\zeta)} \xi^\top Q y, \qquad (\text{P})$$

where $Q \in \mathbb{R}_+^{n \times n}$, W is defined as in $\Gamma-$SPD, Y is any bounded subset of $\mathbb{R}^n$, and Ξ_{knap} is a knapsack uncertainty set [3]. If the nominal counterpart of P can be solved in polynomial time, it is possible to solve P in polynomial time when b is constant.

References

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