Algorithmic aspects of quasi-kernels

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1 Introduction

Let $D = (V, A)$ be a digraph. A *kernel* K is a subset of vertices that is independent (*i.e.*, all pairs of distinct vertices of K are non-adjacent) and such that, for every vertex $v \notin K$, there exists $w \in K$ with $(v, w) \in A$. Kernels were introduced in 1947 by von Neumann and Morgenstern [9]. It is now a central notion in graph theory and has important applications in relations with colorings, perfect graphs, game theory and economics, logic, etc. Chvátal proved that deciding whether a digraph has a kernel is NP-complete [2] and the problem is equally hard for planar digraphs with bounded degree [4].

Chvátal and Lovász [1] introduced quasi-kernels in 1974. A *quasi-kernel* in a digraph is a subset Q of vertices that is independent and such that every vertex of the digraph can reach some vertex in Q via a directed path of length at most two. In particular, any kernel is a quasi-kernel. Yet, unlike what happens for kernels, every digraph has a quasi-kernel. Chvátal and Lovász provided a proof of this fact, which can be turned into an easy polynomial-time algorithm.

In 1976, Erdős and Székely [3] conjectured that every sink-free digraph $D = (V, A)$ has a quasi-kernel of size at most $|V|/2$. This question is known as the *small quasi-kernel conjecture*. So far, this conjecture is only confirmed for narrow classes of digraphs. In 2008, Heard and Huang [6] showed that every digraph D has two disjoint quasi-kernels if D is a sink-free tournament or a transitive digraph. In particular those graphs respect the small quasi-kernel conjecture. Recently, Kostochka et al. [7] renewed the interest in the small quasi-kernel conjecture and proved that the conjecture holds for orientations of 4-colorable graphs (in particular, for planar graphs).

2 Disjoint quasi-kernels

Towards proving the small quasi-kernel conjecture, Gutin et al. conjectured in 2001 that every sink-free digraph has two disjoint quasi-kernels, which would imply the small quasi-kernel conjecture. The same authors constructed a nice counterexample with 14 vertices [5] in 2004. We show that, not only sink-free digraphs occasionally fail to contain two disjoint quasi-kernels, but it is NP-complete to distinguish those that do from those that do not (our proof uses the counterexample constructed by Gutin et al.).

Theorem 2.1. *Deciding if a digraph has two disjoint quasi-kernels is* NP*-complete, even for digraphs with maximum out-degree six.*

Whereas the small quasi-kernel conjecture is true for sink-free planar digraphs [7], no sink-free planar digraph without two disjoint quasi-kernels is known so far. The sink-free digraph constructed in Theorem 2.1 is not planar as it uses the counterexample constructed by Gutin et al. [5] that contains an orientation of K_7. This raises the question of deciding whether every sink-free planar digraph has two disjoint quasi-kernels. Although we have not been able to answer to this question, we show that it is NP-complete to distinguish those sink-free planar digraphs that have three disjoint quasi-kernels from those that do not.

Proposition 2.2. *Deciding if a digraph has three disjoint quasi-kernels is* NP*-complete, even for bounded degree planar digraphs.*

3 Minimum size quasi-kernels

In this work we also initiate the study of the problem of finding the minimum size of a quasi-kernel which we call MinQK (and we let QK stand for the related decision problem).

Proposition 3.1. *MinQK is polynomial-time solvable for orientations of trees.*

The next proposition shows that there is not so much room for extending Proposition 3.1.

Proposition 3.2. *QK is* NP*-complete, even for acyclic orientations of cubic graphs.*

Assuming $\mathsf{FPT} \neq \mathsf{W}[2]$, we show that one cannot confine the seemingly inevitable combinatorial explosion of computational difficulty to an additive function of the size of the quasi-kernel.

Proposition 3.3. *QK is* W[2]*-complete for parameter the size of the quasi-kernel, even for acyclic orientations of bipartite graphs.*

Combining the inapproximability result for Set Cover by Raz and Safra [8] with the proof of Proposition 3.3, we obtain the following result.

Proposition 3.4. *MinQK cannot be approximated in polynomial time within a factor of $c \ln(|V|)$ for some constant c unless $\mathsf{P} = \mathsf{NP}$, even for acyclic orientations of bipartite graphs.*

As for non-approximability, we have the following result.

Proposition 3.5. *MinQK is* APX*-complete for acyclic digraphs with maximum in-degree three.*

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