Intersection Cuts for Mixed-Integer Signomial Sets

Liding Xu[1]

OptimiX, LIX CNRS, École Polytechnique, Institut Polytechnique de Paris, France
liding.xu@polytechnique.edu

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1 Introduction

Intersection cuts [3, 10] have been tackled in recent years to tighten the polyhedral outer approximation of certain non-convex sets such as lattice set [1], bi-level set [5], outer product set [2] and quadratic set [9].

In this work, we study the intersection cuts for *signomial* set and *mixed-integer-signomial* set. Signomial Programs (SPs) and Mixed-Integer Signomial Programs (MISPs) [6, 8] contain signomial terms, the intersection cuts can tighten their Linear Programming (LP) relaxations.

Given an $n-$dimensional multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbf{R}^n$, a signomial term is defined as $x^\alpha := \prod_{i=1}^n x_i^{\alpha_i}$. We assume the variables being positive, thus allowing the multi-index α to have negative components. We define the signomial set as follows :

$$\mathcal{S}_s := \{(x, y) \in \mathbf{R}^n_+ \times \mathbf{R} : y \,(\cdot)\, x^\alpha\},$$

where $\cdot$ can be $\leq, =$ or $\geq$.

We define the mixed-integer counterpart of the signomial set. Let $[\underline{x}, \overline{x}]$ be the box constraint on x such that $\underline{x} > 0$ implicitly imposes the positiveness of x. W.l.o.g., we assume the first $n_1 \leq n$ entries of x to be integer, and let $\mathcal{D} = (\mathbf{Z}^{n_1}_+ \times \mathbf{R}^{n-n_1}_+) \cap [\underline{x}, \overline{x}]$ be the bounded mixed-integer domain. Similarly, the mixed-integer-signomial set is defined as :

$$\mathcal{S}_{mis} := \{(x, y) \in \mathcal{D} \times \mathbf{R} : y \,(\cdot)\, x^\alpha\}.$$

For a non-convex set $\mathcal{S}$, a convex set $\mathcal{C}$ is $\mathcal{S}-$free if $\operatorname{int} \mathcal{C} \cap \mathcal{S} = \emptyset$. The intersection cut framework requires two ingredients to generate a valid inequality to separate $\bar{z}$:

i) a translated simplicial conic relaxation $\mathcal{R}$ with vertex $\bar{z}$ such that $\mathcal{S} \subset \mathcal{R}$ and $\bar{z} \notin \mathcal{S}$;

ii) a $\mathcal{S}-$free set $\mathcal{C}$ such that $\bar{z} \in \operatorname{int} \mathcal{C}$.

The spatial-branch-and-bound (sBB) algorithm [7] is a general technique to find the global solution of non-convex MINLPs. The second ingredient of intersection cuts, i.e., recession conic relaxation, can be retrieved from the LP-based sBB algorithm. If two $\mathcal{S}-$free sets $\mathcal{C}_1, \mathcal{C}_2$ are such that $\mathcal{C}_1 \subset \mathcal{C}_2$, the intersection cut derived from $\mathcal{C}_2$ dominates the one derived from $\mathcal{C}_1$ [4]. Therefore, the maximal $\mathcal{S}-$free sets are ideal and crucial to generate strong intersection cuts.

The set $\mathcal{S}$ has a *difference-of-concave* (DC) representation if it can be expressed as

$$\mathcal{S} := \{z \in \mathbf{R}^p : f_1(z) - f_2(z) \leq 0\},$$

where f_1 and f_2 are concave functions.

[10] proposed the *linearization* technique to derive $\mathcal{S}-$free sets in DC-representations. The linearization technique constructs the following $\mathcal{S}-$free set :

$$\mathcal{C} := \{z \in \mathbf{R}^p : f_1(z) - (f_2(\tilde{z}) + \nabla f_2(\tilde{z}) \cdot (z - \tilde{z})) \geq 0\},$$

where $f_2(\tilde{z}) + \nabla f_2(\tilde{z}) \cdot (z - \tilde{z})$ is the linearization (over-estimator) of f_2 at $\tilde{z}$.

Indeed, a non-convex set $\mathcal{S}$ may have various DC-representations. An *ideal* DC-representation should yield maximal $\mathcal{S}-$free sets using the linearization technique.

2 Our results

We show that the signomial set can be transformed into the following *normalized* DC-representation :

$$\mathcal{S}_s = \{(u, v) \in \mathbf{R}^h_+ \times \mathbf{R}^l_+ : u^\beta - v^\gamma \leq 0\}, \tag{1}$$

where $\beta \in \mathbf{R}^h_+$ and $\gamma \in \mathbf{R}^l_+$ are $h-$ and $l-$ dimensional multi-indices such that $\max\{\|\beta\|_1, \|\gamma\|_1\} = 1$. Because u^β, v^γ are concave, the representation yields a $\mathcal{S}_s-$free set using the linearization technique.

For any non-convex set $\mathcal{S}$, we give a sufficient condition to guarantee the maximality of a $\mathcal{S}-$free set ; then, we show that the normalized DC-representation is *ideal* in the sense that it yields the maximal $\mathcal{S}_s-$free set in the non-negative orthant. Some non-ideal DC-representations of $\mathcal{S}_s$ are given as well.

We propose a method to enlarge the $\mathcal{S}_s$-free set into the mixed-integer-signomial-free ($\mathcal{S}_{mis}-$free) set, where both nonlinear constraints and bounded/integer variables are featured. A byproduct is a strong convex relaxation of the following mixed-integer power cone :

$$\{(x, t) \in \mathcal{D} \times \mathbf{R} : x^\alpha \geq t\}, \tag{2}$$

where $\alpha \in \mathbf{R}^p_+, \|\alpha\|_1 \leq 1$, and $\mathcal{D} \subset \mathbf{R}^p_+$ is a bounded mixed-integer domain. The relaxation is stronger than the continuous relaxation $\{(x, t) \in \mathbf{R}^p_+ \times \mathbf{R} : x^\alpha \geq t\}$.

The intersection cut separation problem is reduced to univariate root-finding problems. Such root-finding problems can be solved numerically in quadratic convergence rate by Newton-like algorithms.

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