An efficient Benders decomposition for the p-median problem

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Keywords : *discrete location, p-median problem, Benders decomposition, polynomial separation algorithm*

1 Introduction

The p-median problem (pMP) is an important discrete location problem in which a given number p of locations have to be chosen from the set of candidate sites. More formally, given a set of N clients $\{C_1, ..., C_N\}$ and a set of M potential sites to open $\{F_1, ..., F_M\}$, let d_{ij} be the distance between client C_i and site F_j and $p \in \mathbb{N}$ the number of sites to open. The objective is to find a set S of p sites such that the sum of the distances between each client and its closest site in S is minimized. The (pMP) leads to applications where the sites may correspond to warehouses, plants, or shelters for example. More recent applications can also be found in clustering processes in databases, where sub-groups of objects, variables, persons, etc. are identified according to defined criteria.

A great interest in solving large location problems has led to the development of various heuristics and meta-heuristics in the literature. However, the exact resolution of large instances remains a challenge [8]. Some location problems have recently been efficiently solved using the Benders decomposition method (see e.g., [4, 6]). In this work, we explore a Benders decomposition of a recent formulation of the (pMP). We prove that the Benders cuts can be separated by a polynomial time algorithm. We implement a *branch-and-Benders-cut* approach that outperforms state-of-the-art methods [1, 7] on benchmark instances.

2 Benders decomposition for the *(pMP)*

The Benders decomposition splits the optimization problem into a *master problem* and one or several *sub-problems*. The master problem and the sub-problems are solved iteratively and at each iteration, each sub-problem may add a cut to the master problem.

Our Benders decomposition is based on the most efficient formulation in the literature [5]. We show that there is a finite number of Benders cuts which lead to a new compact formulation for the p-median problem. The efficiency of our decomposition comes from a polynomial algorithm for the resolution of the sub-problems in conjunction with several implementation improvements in a two-phase resolution.

In Phase 1, the integrity constraints are relaxed, which allows many cuts to be generated quickly. In order to enhance the performance of this phase, we note that providing a good candidate solution to the initial master problem can significantly reduce the number of iterations. Consequently, we use state-of-art heuristic [10] as an input of our decomposition algorithm. Additionally, we use a rounding heuristic at each iteration to try to improve the upper bound of the problem.

At the end of Phase 1, most of the generated Benders cuts are not saturated by the current fractional solution. We remove most of them to reduce the number of constraints in the master problem. Phase 1 provides a lower and an upper bound of the problem, which can be used to perform an analysis of the reduced costs of the last fractional solution to also reduce the number of variables.

In the Phase 2, we add the integrity constraints and the obtained master problem is solved through a *branch-and-Benders-cut*. At each node which provides an integer solution, we solve the sub-problems in order to generate Benders cuts. The resolution of the sub-problems is performed through callbacks which is a feature provided by mixed integer programming solvers.

3 Results

We present our results on about 200 benchmark instances of different sizes with a limit of 10 hours of CPU time. We consider the same instances used in the stat-of-art methods [1, 7] from OR-Library [2] and TSP-Library [9] satisfying triangle inequalities. We show that our approach provides betters results that [1, 7] and we can also provide for the first time the optimal values for 7 instances having up to 115.475 clients and sites. The state-of-the-art method in [7] could solve a few instances with up to 85.900 clients and sites.

We also tested our decomposition on other p-median instances: the RW instances [10] which are randomly generated instances that do not satisfy triangular inequalities and the Optimal Diversity Management instances (ODM) [3] in which there are allocation prohibitions between certain customers and sites. RW instances have not been handled by exact methods and the method in [7] does not support ODM instances. We were able to solve RW instances of up to 1000 clients with a large p value and all ODM instances with 3773 clients and sites, within our time limit of 10 hours.

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