Optimization problems in graphs with locational uncertainty

Marin Bougeret[1], Jérémy Omer[2], Michael Poss[2]

[1] LIRMM, University of Montpellier, CNRS, France
{marin.bougeret,michael.poss}@lirmm.fr
[2] IRMAR, INSA de Rennes, Rennes, France
jeremy.omer@insa-rennes.fr

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1 Introduction

Research in combinatorial optimization has provided efficient algorithms to solve a large variety of complex discrete decision problems, providing exact or near-optimal solutions in reasonable amounts of time. The applications are countless, ranging from logistics (network design, facility location, ...) to scheduling, including even important data science applications such as clustering. Many of these applications amount to select a subset of edges of a graph $G = (V, E)$ among a family of feasible subsets $\mathcal{F}$ and that minimizes its total weight. Among those, we focus on spatial graphs on a given metric space $(\mathcal{M}, d)$, where each vertex i is assigned a position $u_i \in \mathcal{M}$ and the cost of set $F \in \mathcal{F}$ is given by $\sum_{\{i,j\}\in F} d(u_i, u_j)$, leading to the combinatorial optimization problem

$$\min_{F\in\mathcal{F}} \sum_{\{i,j\}\in F} d(u_i, u_j). \tag{1}$$

Problem (1) encompasses many applications, such as network design, facility location, and clustering. These are typically subject to data uncertainty, be it because of the duration of the decision process, measurement errors, or simply lack of information.

One successful framework that has emerged to address uncertainty is robust optimization [2], modeling the uncertain parameters with convex sets, such as polytopes, or finite sets of points, among which combinatorial robust optimization focuses on discrete robust optimization problems [4]. We enter this framework by considering the model where the positions of the vertices are subject to uncertainty, therefore impacting the distances among the vertices. The resulting problem thus seeks to find the feasible subgraph that minimizes its worst-case sum of distances. Formally, we introduce for each vertex $i \in V$ the set of possible locations as the uncertainty set $\mathcal{U}_i \subseteq \mathcal{M}$. Using the notations $u = (u_1, \ldots, u_{|V|})$ and $\mathcal{U} = \times_{i\in V}\mathcal{U}_i$, the general problem considered in this paper can be cast as

$$\min_{F\in\mathcal{F}} \max_{u\in\mathcal{U}} \sum_{\{i,j\}\in F} d(u_i, u_j). \tag{2}$$

2 Literature

Traditionally, robust optimization problems with an objective function that is concave in the uncertain parameters are reformulated as monolithic models using conic duality [2]. These techniques do not readily extend to function $d(u_i, u_j)$ as the latter is non-concave in general. Actually, for Euclidean metric spaces based on the vector space $\mathbb{R}^\ell, \ell \in \mathbb{Z}^+$, $d(u_i, u_j) = \|u_i - u_j\|_2$ is convex in u_i and u_j. Function $\|u_i - u_j\|_2$ is closely related to the second-order cone

(SOC) constraints considered by [5] for robust problems with polyhedral uncertainty sets. The authors of [5] linearize such robust SOC constraints by introducing adjustable variables, turning the problem into an adjustable robust optimization problem.

A second work closely related to (2) is [3], which relies on computational geometry techniques to provide constant-factor approximation algorithms in the special case where $\mathcal{F}$ contains all Hamiltonian cycles of G. They propose in particular to solve a deterministic counterpart of (2) where the uncertain distances are replaced by the maximum pairwise distances $d_{ij}^{max} = \max_{u_i \in \mathcal{U}_i, u_j \in \mathcal{U}_j} d(u_i, u_j)$, for each $(i, j) \in V^2, i \neq j$.

3 Contributions

Our contributions can be summarized as follows :

— We prove that problem (2) is $\mathcal{NP}$-hard even when $\mathcal{F}$ consists of all $s - t$ paths and $(\mathcal{M}, d)$ is the one-dimensional Euclidean metric space or when $\mathcal{F}$ consists of all spanning trees of G. These results illustrate how the nature of problem (2) fundamentally differs from the classical min-max robust problem with cost uncertainty, which is known to be polynomially solvable whenever the costs lie in independent uncertainty sets [1].

— We provide a general cutting-plane algorithm for problem (2) that relies on integer programming formulations for $\mathcal{F}$. We further show that the separation problem $c(F) = \max_{u \in \mathcal{U}} \sum_{\{i,j\} \in F} d(u_i, u_j)$ is $\mathcal{NP}$-hard and provide two algorithms for computing $c(F)$. One is based on integer programming formulations while the other one relies on a dynamic programming algorithm that involves the threewidth of F.

— We extend the approximation algorithm based on d^{max} to general sets $\mathcal{F}$ and metric spaces different from the Euclidean one. We study in depth the resulting approximation ratios, which depend on the structure of $\mathcal{F}$ and $(\mathcal{M}, d)$.

— We provide a dynamic programming algorithm for the special case where $\mathcal{F}$ consists of all $s - t$ paths, which is turned into a fully-polynomial time approximation scheme by rounding data appropriately.

— We compare numerically the exact cutting plane algorithm with the approximation algorithm that relies on d^{max}. The benchmark is composed of two families of instances. The first family includes Steiner tree instances that illustrate subway network design. The second one is composed of strategic facility location instances. The former application relies on two-dimensional Euclidean metric spaces so we can further include the affine decision rule reformulation from [5] to the comparison.

Références

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