A branch-and-cut-and-price algorithm for the connected max-k-cut problem

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1 Problem

The Connected Max-k-Cut Problem (CMkCP) is defined on an undirected weighted graph $G = (V, E)$ with for each $e \in E$ an edge weight c_e. The goal is to partition the graph into K clusters such that each cluster forms a connected subgraph. The objective is to maximize the sum of the weights of the edges between the clusters. A recent branch-and-cut algorithm have been proposed in [3] and a slight variation in [2]. The later authors also investigate the use of a column generation approach [1]. The main differences with the work presented here are the model of the subproblem as well as this work incorporates more techniques such as stabilization and cuts at the master level.

2 Master problem

Let Ω be the set of all connected subgraphs (cluster) of G. A subgraph $i \in \Omega$ has an inter cost c_i that is equal to the sum of the edges in the co-cycle induced by the cluster. For a cluster S, its co-cycle is $\omega(S) = \{\{i, j\} \in E : (i \in S \wedge j \notin S) \vee (i \notin S \wedge j \in S)\}$. A binary constant a_{ij} is equal to 1 f f and only if the vertex j belongs to the cluster i. Let's θ_i be a binary variable equal to 1 if and only if $i \in \Omega$ is selected. The master problem is then :

$$\text{Maximize } \sum_{i \in \Omega} c_i \theta_i, \tag{1}$$

subject to

$$\begin{aligned}
\sum_{i \in \Omega} a_{ij} \theta_i &= 1 \qquad (j \in V), & (2)\\
\sum_{i \in \Omega} \theta_i &= K, & (3)\\
\theta_i &\in \{0, 1\} \qquad (i \in \Omega). & (4)
\end{aligned}$$

A stabilization approach based on the box stabilization technique of Pigatti et al. [6]. The lower bound on the master problem is reinforced by means of subset row cuts (SRC) [5]. More precisely, 3-SRC are used.

3 Subproblem

The subproblem is defined on G with an additional profit associated to each $j \in V$ equal to π_j. The problem consists in searching for a connected subgraph $G[S]$ induced by a subset

$S \subseteq V$. The reduced cost of a cluster defined by S is $c(S) = \sum_{e \in \omega(S)} c_e - \sum_{j \in S} \pi_i - \pi_0$. We are looking for a set S with a positive reduced cost. Different integer programs have been tested. The most efficient is an integer program presented below that extends the model for the Prize-Collecting Steiner problem of Ljubic et al. [4]. Additional constraints are also investigated to speed-up the solution of the subproblem.

$$\text{Maximize} \sum_{e \in E} c_e z_e - \sum_{(i,j) \in A} \pi_i x_{ij} - \sum_{i \in V} \pi_0 x_{ri} \tag{5}$$

subject to

$$\sum_{j \in \delta_i} x_{ji} = y_i \quad (i \in V) \tag{6}$$

$$\sum_{i \in V} x_{ri} = 1 \tag{7}$$

$$\sum_{(i,j) \in \omega^-(S)} x_{ij} \geq y_k \quad (S \subset V', k \in S, r \notin S) \tag{8}$$

$$z_e + y_i + y_j \leq 2 \quad (e = \{i,j\} \in E) \tag{9}$$

$$z_e - y_i - y_j \leq 0 \quad (e = \{i,j\} \in E) \tag{10}$$

$$y_i \in \{0,1\} \quad (i \in V) \tag{11}$$

$$x_{ij} \in \{0,1\} \quad ((i,j) \in A) \tag{12}$$

$$z_e \in [0,1] \quad (e \in E) \tag{13}$$

3.1 Conclusions and current works

The master problem and the subproblem are embedded in a branch-and-cut-and-price algorithms. The branching strategy is classic and it can be easily considered in the subproblem by means of inequalities. Experiments show that the use of the proposed model for the subproblem strictly outperforms the one proposed in [1]. Current experiments explore the impact of the additional constraints on the solution of the subproblem and the comparison of the complete algorithms with the polyhedral approaches from the literature. The development of a dynamic algorithm to solve the subproblem is also under investigation.

Références

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