A Robust version of the Ring Star Problem

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We consider the Ring Star network design, where we are given a complete mixed graph with both arcs from and to every node, as well as edges between any pair of different nodes and a given node called depot. The Ring Star Problem (RSP) is a NP-hard problem that consists in selecting a subset of nodes including the depot, named hubs, and link them with a cycle to form the ring. Then each non-hub node, called terminal, is connected to exactly one hub in the cycle, which is the star topology part. The aim of RSP is to minimize the sum of three costs corresponding to (i) selecting the subsets of hubs, (ii) linking the ring, and (iii) connecting the star.

RSP has been widely studied in the literature, Labbé *et al.* [3] proposed a Mixed Integer Programming model, strengthened with valid inequalities studied with a polyhedral analysis and solved with a Branch-and-Cut algorithm. Earlier, Xu *et al.* [5] solved with a tabu search a particular case of RSP where hubs and terminals have to be selected from distinct subsets. An exact solution approach that takes advantage of the fact that the depot must be in the ring can be found in Kedad-Sidhoum and Nguyen [2]. Calvete *et al.* [1] have presented an evolutionary-based heuristic for solving RSP while Zang *et al.* [6] recently introduced an ant colony system algorithm.

In this paper, we study the Robust Ring Star Problem, referred to as ρ-RSP. Let V be the node set of the input graph, ρ-RSP has an additional input compared to RSP: $\widetilde{V} \subseteq V$ is a possibly empty subset of nodes that can fail if they are selected as hubs. A hub in $\widetilde{V}$ is called *uncertain* because it may fail, whereas a hub in $V \backslash \widetilde{V}$ is called *certain* as it is not supposed to fail [4].

The ρ-RSP is to build a minimum cost subgraph of G that will always contain a ring-star topology even if a single hub in $\widetilde{V}$ fails. By contrast with RSP, if a hub belongs to $\widetilde{V}$, an additional edge has to join its two neighbors in the ring. This edge will be used if the hub in question fails. Furthermore, each terminal is linked to the ring either by a single hub in $V \backslash \widetilde{V}$ (if there exist one), or by connecting it to two hubs in $\widetilde{V}$. The ρ-RSP is then to design a minimum cost robust ring-star network. Thus, it can be observed that ρ-RSP reduces to RSP when $\widetilde{V}$ is empty. Figure 1 shows an illustration of the ρ-RSP with $\widetilde{V} = V \setminus \{1, 5, 6\}$ where hubs are shown in red, and terminals are shown in blue. It can be seen that terminal 10 is connected to two hubs, because they are uncertain.

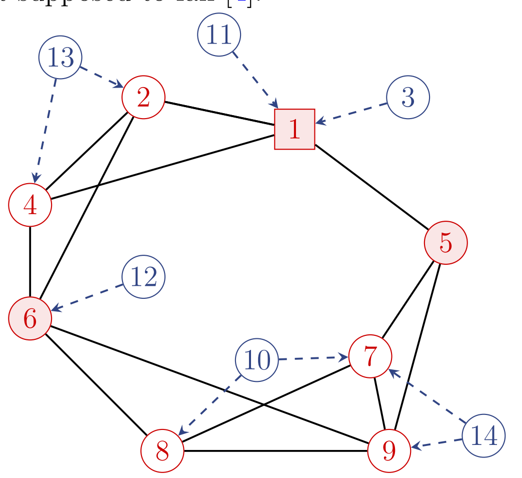

FIG. 1: An instance of the ρ-RSP with $\widetilde{V} = V \backslash \{1, 5, 6\}$.

To the best of our knowledge, the ρ-RSP is a new problem in the literature. We have currently formulated and implemented an Integer Linear Programming (ILP) model for ρ-RSP. We also present a Benders decomposition that is composed by, first, a Master problem that determines which nodes will be selected as hubs then link them with a cycle and, second, a subproblem that links terminals to the ring and connects the two neighbors in the ring of all uncertain hubs. Both the master problem and the subproblem are formulated as Integer Linear Programs.

Let the cost for connecting hubs $i \in V$ and $j \in V \setminus \{i\}$ be r_{ij} and the cost for linking terminal i to hub j be s_{ij}. $\hat{x}_{ij}$ and $\hat{y}_{jj}$ are master problem's fixed variables, $\hat{y}_{jj}$ is set to one if node j is selected to be a hub while $\hat{x}_{ij}$ is set to one if both hubs i and j are connected in the master problem. Variables x'_{ij} are one if an edge connects hubs i and j and y_{ij} is one if terminal i is connected to hub j and the subproblem's objective is to minimize the cost of the edges that should be added to cope with the failure of uncertain hubs, plus the total star cost. The subproblem's primal is stated as follows:

$$\text{Minimize } \lambda = \sum_{i \in V} \sum_{\substack{j \in V \\ i<j}} r_{ij} x'_{ij} + \sum_{i \in V} \sum_{j \in V \setminus \{i\}} s_{ij} y_{ij}$$

$$\sum_{\substack{j \in V \setminus \widetilde{V} \\ i \neq j}} 2y_{ij} + \sum_{\substack{j \in \widetilde{V} \\ i \neq j}} y_{ij} = 2(1 - \hat{y}_{ii}) \qquad \forall i \in V \tag{1}$$

$$x'_{ik} \geq \hat{x}_{ij} + \hat{x}_{jk} - 1 \quad \forall (i,j,k) \in V^3 : j \in \widetilde{V}, i \neq j, j \neq k, i < k \tag{2}$$

$$y_{ij} \leq \hat{y}_{jj} \qquad \forall (i,j) \in V, i \neq j \tag{3}$$

$$y_{ij} \in \{0,1\} \qquad \forall (i,j) \in V, i \neq j \tag{4}$$

$$x'_{ij} \in \{0,1\} \qquad \forall (i,j) \in V^2, i < j$$

Constraints (1) enforce that each terminal is connected to exactly two distinct hubs if these hubs are in $\widetilde{V}$, or to a single hub if it is in $V \setminus \widetilde{V}$. Constraints (2) enforce that for each hub $j \in \widetilde{V}$ having hubs i and k as neighbors in the ring, there is an edge that joins i and k, that can serve when j fails. There are $\frac{1}{2}\tilde{n}(n-1)(n-2) = \mathcal{O}(\tilde{n}n^2)$ such inequalities where $n = |V|$ and $\tilde{n} = |\widetilde{V}|$. Constraints (3) ensure that a terminal can only be linked to a hub. We propose to add a class of valid inequalities to the subproblem's primal, which guaranty that the continuous relaxation of the subproblem has an integral solution. The subproblem, originally an integer linear program, can be stated as a linear program by adding the following constraint: $y_{ik} \leq \sum_{j \in \widetilde{V} \setminus \{k\} : \hat{y}_{jj}=1} y_{ij}$ for all $i \in V$ such that $\hat{y}_{ii} = 0$, and for all $k \in \widetilde{V}$ such that $s_{ik} = \min_{j \in \widetilde{V} : \hat{y}_{jj}=1} s_{ij}$. If no such k exists, the constraint is not enforced. This constraint states that if terminal i is connected to an uncertain hub, then it should be connected to at least another one. Since (3) dominates (4), and because the objective function "pushes" the x' variables downward, integrality constraints can be dropped, and it can be shown that the linear relaxation of the subproblem (with the new constraints above) has an integral optimal solution. This leads us to introduce a quadratic time algorithm for addressing the dual of the continuous relaxation of the subproblem, which accelerates the solution process of the Benders decomposition.

Numerical experiments are carried out to compare the results of the Integer Linear Programming formulation and of the Benders decomposition. We observed that Benders decomposition currently outperforms the ILP in instances where the ring is small, i.e. when there are few hubs.

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