Multistage stochastic programs with the entropic risk measure

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1 Introduction

In order to make sequential decisions under uncertainty, it is convenient to consider *risk measures*. Most works use dynamic programming, and consider the so-called *nested* risk measures. [2, 3]. When we care about the outcome of our decision at the last stage, *end-of-horizon* formulations are more appropriate. In this work we study the entropic risk measure, for which the two approaches coincide.

2 Conditional consistency and the entropic risk measure

Consider a sequence of correlated real-valued random variables, $Z = \{Z_t\}_{t=1}^T$. We have

$$\begin{aligned} End-of-Horizon-Risk(Z) &= \mathbb{F}[Z_1 + Z_2 + \cdots + Z_T], \\ Nested-Risk(Z) &= \mathbb{F}\left[Z_1 + \mathbb{F}_{Z_2|Z_1}[Z_2 + \mathbb{F}_{Z_3|Z_2}[\cdots + \mathbb{F}_{Z_T|Z_{T-1}}[Z_T]\right]. \end{aligned}$$

Définition 1 *Let (X_1, X_2) and (Y_1, Y_2) be two-dimensional vectors for which requisite expectations are finite. A risk measure $\mathbb{F}$ is said to be conditionally consistent if :*

$$\mathbb{F}[X_1 + X_2] \leq \mathbb{F}[Y_1 + Y_2] \iff \mathbb{F}[X_1 + \mathbb{F}_{X_2|X_1}[X_2]] \leq \mathbb{F}[Y_1 + \mathbb{F}_{Y_2|Y_1}[Y_2]].$$

An example of a convex risk measure that is conditionally consistent is the entropic ($\gamma > 0$) :

$$\mathbb{ENT}_\gamma[Z] = \frac{1}{\gamma} \log\left(\mathbb{E}[e^{\gamma Z}]\right). \quad (1)$$

2.1 Risk-averse multistage Stochastic Programming (MSSP)

We consider a T-stage MSSP given by

$$\begin{aligned} V_t(x_{t-1}, \omega_t) = \min_{\bar{x}_t, x_t \geq 0} \quad & c_t^\top x_t + \mathbb{F}_{\omega_{t+1} \in \Omega_{t+1}}[V_{t+1}(x_t, \omega_{t+1})] \\ & \bar{x}_t = x_{t-1} \quad [\lambda] \\ & A_t x_t + B_t \bar{x}_t = b_t. \end{aligned} \quad (2)$$

We can form single-cut master problems that approximate the stage-wise problems as follows :

$$\begin{aligned} V_t^K(x_{t-1}, \omega_t) = \min_{\bar{x}_t, x_t \geq 0, \theta_{t+1} \geq -M_{t+1}} \quad & c_t^\top x_t + \theta_{t+1} \\ & \bar{x}_t = x_{t-1} \quad [\lambda] \\ & A_t x_t + B_t \bar{x}_t = b_t \\ & \theta_{t+1} \geq \alpha_{t+1,k} + \beta_{t+1,k}^\top x_t, \quad k = 1, \ldots, K-1, \end{aligned} \quad (3)$$

where M_{t+1} is a lower bound on $\mathbb{F}[V_{t+1}(\cdot, \omega_{t+1})]$. Explicit values of α_{t+1} and β_{t+1} can be computed for the entropic risk measure, and such formulation is amenable to SDDP.

3 Numerical illustration

Consider a road network with three arcs (see [1]) as shown in Figure 1. Travel times are independent, and we must choose between $\mathbb{F}[X+Z]$ and $\mathbb{F}[Y+Z]$.

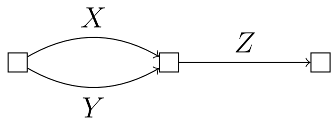

FIG. 1 – Road network with three independent travel times, X, Y, and Z.

The results are shown in Figure 2 : for the $\mathbb{CV@R}$, when $\gamma \in (0.865, 0.97)$, the optimal decision to the end-of-horizon model is to switch back to X. In Figure 2b, since the end-of-horizon and nested formulations are equivalent, there is only one line visible on the graph.

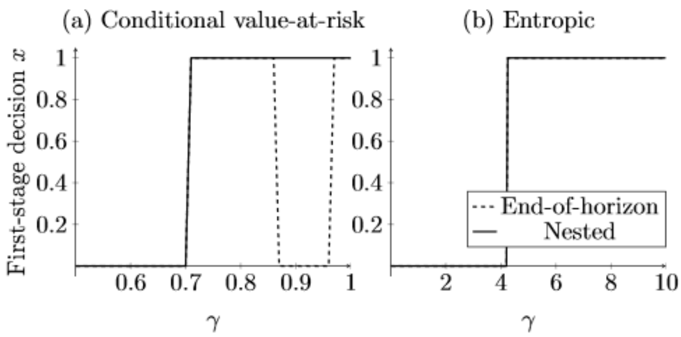

FIG. 2 – $x=0$ (road Y), $x=1$ (road X) *against*γ using $\mathbb{CV@R}$ (a) and entropic (b)

4 Conclusions

We studied the entropic risk measure in MSSP. We showed that it can be embedded within decomposition algorithms such as the SDPP, and exemplified its use in a transportation problem.

Références

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